Mergers and Self-Enforcing Constraints

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1 Introduction

Why do firms merge? Two strands of the economic literature try to answer this question. Since the beginning of the 80's, Industrial Organization economists have tried to find a simple model to explain why firms merge. In their paper, Salant *et al* [17] show that under a quantity-competition framework, unless synergies are important or a majority of firms are involved (more than 80 percent of firms), merged firms (insiders) lose while other firms (outsiders) gain.[1] Deneckere and Davidson [3] state clearly the problem.

> The incentive to merge in noncooperative oligopoly models depends on the interaction of two basic forces. First, a merger allows coalition partners to absorb a negative externality. (...) Second, the merger elicits a spiral of responses from rival firms. (...) In quantity-setting games, (...) the response of other industry members tends to hurt coalition partners because in these games reaction functions are typically downward sloping.[2]

Some authors have proposed alternative approaches. Kamien and Zang [11] present a three-stage model. The first stage is the acquisition phase where firms bid to acquire other firms. In the second stage, merged firms (the parent firm) decide how many divisions (old independent firms) will produce a strictly positive quantity of goods. In the last stage, divisions of every parent firm compete in a Cournot game. This approach differs from the Salant *et al* [17] model. Implicitly, Salant *et al* [17] assume that all firms involved in a merger act post-merger as a unique entity. With their model, Kamien and Zang [11] find that 50 percent of market firms must be involved in the merger to gain from the merger. Creane and Davidson [2][3] continue in the same way and propose a model in which the parent firm can use a different strategy with their divisions. They show that the merger could be beneficial if the parent firm uses a structure in which divisions announce sequentially the quantity they will produce. This Stackelberg game, which is played by divisions in combination with a Cournot game with the other firms, leave insiders with a gain and outsiders with a loss. Moreover, they find that only a small number of firms must be involved in

[1] Deneckere and Davidson [3] work on a price competition model. They find that both insiders and outsiders gain but outsiders do better than insiders.
[2] Deneckere and Davidson [3], page 484.
[3] Huck, Konrad and Müller [9] present similar models with same results.

the merger. They argue that other kind of strategies can be used to increase the market power of the merging firm. As such, they provide an answer to the merger paradox.[4]

Finance Economists have also studied mergers. They use financial incentives to study conditions under which a merger could be beneficial to insiders.[5] While some authors look at the management incentives,[6] one of the most important approach relates to the optimality of using internal financing versus external financing. In a frictionless capital market framework, Modiglani and Miller [15] show that the capital structure (internal or external financing) of firms does not affect a firm market value. But some economists argue that the equivalence between internal versus external financing does not hold. Alchian [1] and Williamson [20] were the first to argue that headquarters are able to monitor production and effort more effectively than outsiders. Then, mergers could be beneficial if this problem of monitoring leads to an inefficient allocation of capital for pre-merger firms. Gertner *et al* [7] present a model in which headquarters can use the surplus of external capital from given project for financing another project. They argue that this internal capital market increases monitoring incentives, decreases entrepreneurial incentives and redeploys financial assets more efficiently. Stein [18] uses another approach. He supposes that the headquarter is able to enact a winner-picking process which consists of the allocation of the constrained capital to the division which provides a better return. Stein [18] supposes that the headquarters have a better knowledge than outsider investors to allocate more effectively. Consequently, the headquarter is able to reallocate capital as the state of nature is revealed and can reassign capital to the good project from the bad one.

Besides the question of the difference between internal and external capital, the imperfection of the financial market could explain why firms merge. The risk is transferred to the financial market and risk-averse shareholders gain from a decrease in the net revenue variance. When the financial market is not perfect, shareholders can be better off by merging their firm with another. If firms have negatively correlated revenues, the merger will decrease the firm's revenue variance by using an internal financial market. However, if firms have positively correlated revenues, it could happen that the increase in the revenue variance will decrease the effect of the financial market imperfection and leave the merged firm with a net gain.

This paper studies this question. In their paper, Inderst and Müller [10] present a model in which a firm must decide to centralize or decentralize borrowing. With the first option, investors and firms can sign a financial contract which is more efficient than contracts signed when borrowing is decentralized. Implicitly, Inderst and Müller [10] assume that the cost to enforce a contract is quite low. So, the agent must respect the contract in any period. When the cost of enforcing a contract is important and the mobility cost for an agent to quit the contract is quite low, the lack of a binding commitment becomes a problem. Indeed, one agent could have the incentive to break the ex-ante optimal contract after the state of nature is revealed. This problem of commitment in risk-sharing contracts can lead to inefficiencies. To avoid this problem, long term contracts must be self-enforcing, which means that no agent could gain by breaking the contract in all possible contingencies.

[4]Pepall, Richards and Norman [16] define the merger paradox as the difficulty to construct a simple economic model which leaves insiders with a gain even if they do not merge in a monopoly.

[5]Hubbard [8] gives a survey of the literature on financial constraints.

[6]For example, see McNeil, Niehaus and Powers [14].

I use this approach to study in which condition a merger could be beneficial for shareholders. Particularly, I want to study the effects of self-enforcing constraints on the efficiency of mergers. A self-enforcing contract is such that, in all possible states of nature, the firm and the borrower must have an incentive to respect the contract. This approach was first introduced by Thomas and Worrall [19]. In their model, agents agree on signing an insurance contract at time 0. Then, at the beginning of each subsequent period, the state of nature is revealed to both agents. Each agent must decide whether to respect the contract or not. If both of them decide to respect the contract, then the transfer of wealth occurs along the terms specified in the contract. If one decides to break the contract, then no wealth is transferred and it is not possible for the agent to sign another contract in the future. If a given contract, which can be viewed as a series of transfers, is such that in any state of nature and for any period, each agent gains more in respecting the contract than in breaking it, then this contract is said to be self-enforcing.

Since general results are hard to provide, I study the case where utility functions exhibit constant relative risk aversion (CARA). I begin by explicitly solving the self-enforcing contract problem when agents have CARA utility functions and there are two states of nature. From the optimal solution, I am able to draw the Pareto frontier in the context where first-best contracts are feasible and when there is no such feasible contract. Second, I look at the effects of a change in the distribution of the random revenue on the optimal contract. I show that an increase in the variance leads to an increase of the range of the discount factor for which the optimal contract is non trivial. Finally, I find that a merger may or may not be beneficial for merged firms depending on the discount rate and the correlation between firm's revenues.

The paper is divided as follows. In Section 2, I present the model which is then solved explicitly with CARA utility functions in Section 3. I analyze the effect of a change in the variance of revenues in Section 4. In Section 5, I study the benefit of a merger in the self-enforcing context. Section 6 provides concluding remarks.

2 Model

The problem is to design an insurance contract between two infinitely-lived risk-averse agents. I suppose that the state of the economy is *i.i.d.* over the finite set $S = \{1, 2, ..., |S|\}$. The revenue of agent 1 can take values $y_1, ..., y_S$ while agent 2 has a constant revenue $\overline{w}$. By convention, $y_s > y_{s-1}$. I denote by y^t the realization of agent 1's revenue in period t.

The utility functions for agents 1 and 2 are respectively $u(c_t^1)$ and $v(c_t^2)$ where c_t^i is the consumption of agent i in period t. I suppose that the utility functions are twice continuously differentiable and strictly concave. Total consumption must satisfy $c_t^1 + c_t^2 \leq y^t + \overline{w}$ for any $y^t \in \{y_1, y_2, ..., y_S\}$.

Let $h_t = (s_1, s_2, s_3, ..., s_{t-1})$ be the history of realized states of the world at period t. The insurance contract δ consists of a series of transfers which in any given period depend on the history and the current state of the world. Let $b_t(h_t, s)$ be the transfer from agent 1 to agent 2 in period t when the history is h_t and the state of nature at period t is s. The transfer could be positive or negative. Consumption in period t can then be expressed as function of the revenue and the transfer ($c_t^1 = y_s - b_t(h_t, s)$ and $c_t^2 = \overline{w} + b_t(h_t, s)$).

Now, let E_s^t be the operator expectation over s conditional on h_{t-1} and let β be the discount rate. I define $U(\delta; h_t)$ and $V(\delta; h_t)$ as the expected net gain for all periods $t, t+1, t+2, \ldots$ for agents 1 and 2 respectively,

$$U(\delta; h_t) = \mathrm{E}_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[u\left(y_s - b_\tau\left(h_\tau, s\right)\right) - u\left(y_s\right) \right] \right]$$

$$V(\delta; h_t) = \mathrm{E}_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[v\left(\overline{w} + b_\tau\left(h_\tau, s\right)\right) - v\left(\overline{w}\right) \right] \right]$$

An optimal contract is a contract δ such that agent 1 maximizes his expected utility when agent 2 obtains a given level of expected utility. This optimal contract is the solution which maximizes:

$$U(\delta, h_1) = \mathrm{E}_s^1 \left[u\left(y_s - b_1\left(h_1, s\right)\right) - u\left(y_s\right) + \beta U\left(\delta, h_2\right) \right] \tag{1}$$

subject to

$$V(\delta, h_1) = \mathrm{E}_s^1 \left[v\left(\overline{w} + b_1\left(h_1, s\right)\right) - v\left(\overline{w}\right) + \beta V\left(\delta, h_2\right) \right] \geq \overline{V}$$

The solution to the maximization problem (1) is first-best. This contract is such that $u'\left(c_t^1\right) / v'\left(c_t^2\right)$ is constant for all periods t and for all states of nature s.

The first-best contract introduces a potentially large transfer from one agent to the other. In some circumstances, it is conceivable that an agent would prefer reneging on the contract rather than making a transfer to the other agent. If contract enforcement is costly, nothing can prevent an agent from doing so.

I now study this case explicitly. I suppose that each agent can leave the contract at any moment. If an agent leaves the contract, I assume the he remains in autarky forever thereafter. For the contract to hold, each agent must have incentives to respect the contract in every period and for every history. To take this into account, I must add self-enforcing constraints to the problem. The optimal self-enforcing contract is derived by solving

$$MAX \quad U(\delta, h_1) \tag{2}$$

subject to

$$\begin{aligned} V(\delta, h_1) &\geq \overline{V} \\ u\left(y_s - b_\tau\left(h_\tau, s\right)\right) - u\left(y_s\right) + \beta U\left(\delta, h_{\tau+1}\right) &\geq 0 \qquad \tau = 1, 2, \ldots \quad \forall s \in S, \quad \forall h_\tau \\ v\left(\overline{w} + b_\tau\left(h_\tau, s\right)\right) - v\left(\overline{w}\right) + \beta V\left(\delta, h_{\tau+1}\right) &\geq 0 \qquad \tau = 1, 2, \ldots \quad \forall s \in S, \quad \forall h_\tau \end{aligned}$$

The additional constraints state that, in any period and state, each agent must have a non-negative surplus from the relationship.

There always exists a self-enforcing contract. The contract where no transfer is made in any period is trivially self-enforcing. I call this contract the trivial self-enforcing contract (TSEC).

Let $\widetilde{b}_t\left(b_{t-1}, s_{t-1}, s_t\right)$ be the first-best transfer at period t in state s_t when the transfer at period

$t-1$ was b_{t-1} and the state of nature was s_{t-1}. In other words, $\widetilde{b}_t(b_{t-1}; s_{t-1}; s_t)$ is such that

$$\frac{u'\left(y^{t-1}-b_{t-1}\right)}{v'\left(\overline{w}+b_{t-1}\right)}=\frac{u'\left(y_s-\widetilde{b}_t\left(b_{t-1};s_{t-1};s_t\right)\right)}{v'\left(\overline{w}+\widetilde{b}_t\left(b_{t-1};s_{t-1};s_t\right)\right)}$$

Thomas and Worrall [19] show that the optimal contract has the following characterization.

1. For any state of nature s, there exists a non-empty interval $\left[\underline{b_s};\overline{b_s}\right]$ such that $b_t(h_t; s)$ belongs to this interval.
2. For any history h_t and state of nature s,

$$b_t\left(h_t;s\right)=\begin{cases}\underline{b_s} & \text{if } \underline{b_s}>\widetilde{b}_t\left(b_{t-1},s_{t-1},s_t\right)\\ \widetilde{b}_t\left(b_{t-1};s_{t-1};s_t\right) & \text{if } \widetilde{b}_t\left(b_{t-1},s_{t-1};s_t\right)\in\left[\underline{b_s};\overline{b_s}\right]\\ \overline{b_s} & \text{if } \overline{b_s}<\widetilde{b}_t\left(b_{t-1},s_{t-1},s_t\right)\end{cases}\tag{3}$$

The optimal contract is as close as possible to the first-best contract subject to self-enforcing constraints which implicitly define the set of $\underline{b_s}$ and $\overline{b_s}$.

3 CARA utility functions

To be able to solve explicitly (2), I use a specific form of utility functions and add some constraints to the problem structure. In this section, I use a constant absolute risk aversion (CARA) utility function, i.e.

$$\begin{aligned}u\left(c_t^1\left(h_t;s\right)\right)&=-e^{-r\left(c_t^1(h_t,s)\right)}\\ v\left(c_t^2\left(h_t;s\right)\right)&=-e^{-q\left(c_t^2(h_t,s)\right)}\end{aligned}$$

where r and q are respectively the risk aversion parameter of agent 1 and agent 2. With this assumption about the form of the utility function, the problem becomes:

$$MAX\quad E_s^1\left[-e^{-r(y^1-b_1(h_1,s))}+e^{-ry^1}+\beta U\left(\delta,h_2\right)\right]\tag{4}$$

subject to

$$\begin{aligned}E_s^1\left[-e^{-q(\overline{w}+b_1(h_1,s))}+e^{-q\overline{w}}+\beta V\left(\delta;h_2\right)\right]&\geq\overline{V}\\ -e^{-r(y_s-b_\tau(h_\tau,s))}+e^{-ry_s}+\beta E_s^\tau[U\left(\delta;h_{\tau+1}\right)]&\geq 0\qquad \tau=1,2,...\quad\forall s\in S,\quad\forall h_\tau\\ -e^{-q(\overline{w}+b_\tau(h_\tau,s))}+e^{-q\overline{w}}+\beta E_s^\tau[V\left(\delta;h_{\tau+1}\right)]&\geq 0\qquad \tau=1,2,...\quad\forall s\in S,\quad\forall h_\tau\end{aligned}$$

It is possible to characterize first-best contracts using simple manipulations. To do so, I must differentiate (4) without the self-enforcing constraints with respect to two different states of nature at two different periods.

$$\begin{aligned}
\frac{u'\left(y_s - b_t\left(h_t, s\right)\right)}{u'\left(\overline{w} + b_t\left(h_t, s\right)\right)} &= \frac{u'\left(y_z - b_\tau\left(h_\tau, z\right)\right)}{u'\left(\overline{w} + b_\tau\left(h_\tau, z\right)\right)} \\
\frac{re^{-r(y_s - b_t(h_t, s))}}{qe^{-q(\overline{w} + b_t(h_t, s))}} &= \frac{re^{-r(y_z - b_\tau(h_\tau, z))}}{qe^{-q(\overline{w} + b_\tau(h_\tau, z))}} \\
r(y_s - b_t\left(h_t, s\right)) - q(\overline{w} + b_t\left(h_t, s\right)) &= r(y_z - b_\tau\left(h_\tau, z\right)) - q(\overline{w} + b_\tau\left(h_\tau, z\right))
\end{aligned}$$

And I obtain:

$$b_\tau\left(h_\tau, z\right) = b_t\left(h_t, s\right) + \frac{r}{(r+q)}(y_z^\tau - y_s^t) \tag{5}$$

This gives the relation between each possible transfer in each possible state of nature and at every period. Equation (5) tells us that the optimal transfer at a specific period in a specific state of nature is linear in the revenues of both agents. Here, there are optimal contracts for special cases.

- If agent 2 has a random revenue w_s, then the first-best contract is characterized by $b_\tau\left(h_\tau, z\right) = b_t\left(h_t, s\right) + \frac{r}{(r+q)}(y_z - y_s) + \frac{q}{(r+q)}(w_s - w_z)$.
- If agent 1 and agent 2 have the same risk-aversion coefficient ($r = q$), then $b_\tau\left(h_\tau, z\right) = b_t\left(h_t, s\right) + \frac{1}{2}(y_z - y_s)$.
- If agent 2 is risk neutral ($q = 0$), then $b_\tau\left(h_\tau, z\right) = b_t\left(h_t, s\right) + y_z - y_s$.

Throughout the rest of the paper, unless I explicitly suppose something else, I assume that agents have the same risk-aversion coefficient ($q = r$). This facilitates the explicit characterization of the optimal contract.[7]

Also, to be able to explicitly solve the problem, I constrain the number of states of nature to two. With more states, the problem rapidly becomes intractable.

3.1 Conditions for a non-trivial solution

Let's say that a contract δ' is stationary if the transfer in state 1 is b_1' and the transfer in state 2 is b_2', no matter what the history is. The next two lemmas are derived from Propositions 4.1 and 4.2 of Kocherlakota [12].

Lemma 1 *If the optimal contract δ^* is first-best, then δ^* is stationary.*

Proof. If a contract is first-best, then the transition of transfers between states of nature at any period is given by (3). Then, the transfer at period t is $b_1\left(h_t, 1\right) = b_1^*$ if the state of nature is 1 for any history h_t and $b_1\left(h_t, 2\right) = b_2^*$ if the state of nature is 2 for any history h_t. ■

Lemma 2 *If there are only two states of nature, then the optimal contract δ^* for (4) monotonically converges to a stationary contract δ'.*

[7] With different risk-aversion coefficients, I obtain a system of polynomial equations of different degrees.

Proof. Let the optimal contract be δ^*. By definition, the contract δ^* gives the appropriate transfer for any state of nature at period 1. Suppose that transfers at period 1 are given by $b_1^*(h_1, 1)$ and $b_1^*(h_1, 2)$.

Without loss of generality, lets assume the state of nature at period 1 is 1. By (3), $b_1^*(h_1, 1)$ belongs to $[\underline{b_1}, \overline{b_1}]$ and, if the state of nature is the same at period t and $t+1$, then transfers in these periods must be the same (i.e. $b_t^*(h_t, s) = b_{t+1}^*(h_{t+1}, s)$). Then, until the state of nature becomes 2, the transfer stays $b_1^*(h_1, 1)$.

Suppose that the state of nature stays 1 for period 1 to period $t-1$ and becomes 2 at period t. Then $b_t^*(h_t, 2)$ must be equal

- to $\underline{b_2}$ if $\underline{b_2} > \widetilde{b}_t\left(b_{t-1}^*(h_{t-1}, 1), 2\right)$;
- or to $\widetilde{b}_t\left(b_{t-1}^*(h_{t-1}, 1), 2\right)$ if $\widetilde{b}_t\left(b_{t-1}^*(h_{t-1}, 1), 2\right) \in [\underline{b_2}, \overline{b_2}]$;
- or to $\overline{b_2}$ if $\overline{b_2} < \widetilde{b}_t\left(b_{t-1}^*(h_{t-1}, 1), 2\right)$.

In case 2, this means that the contract is first-best and by Lemma 1, the contract is stable.

Suppose case 1, i.e. the transfer in state 2 is the lowest possible ($\underline{b_2}$). If I stay in state 2, then the transfer stays $\underline{b_2}$. If I return to state 1 at period $\tau > t$, then $b_\tau^*(h_\tau, 1)$ must be equal:

- to $\underline{b_1}$ if $\underline{b_1} > \widetilde{b}_\tau\left(\underline{b_2}, 1\right)$;
- or to $\widetilde{b}_\tau\left(\underline{b_2}, 1\right)$ if $\widetilde{b}_\tau\left(\underline{b_2}, 1\right) \in [\underline{b_1}, \overline{b_1}]$;
- or to $\overline{b_1}$ if $\overline{b_1} < \widetilde{b}_\tau\left(\underline{b_2}, 1\right)$.

In case 2, this means the contract become stable after period τ with $\widetilde{b}_\tau\left(\underline{b_2}, 1\right)$ in state 1 and $\underline{b_2}$.

In case 3, this means the contract become stable after period τ with $\overline{b_1}$ in state 1 and $\underline{b_2}$

Case 1 is impossible. I have supposed that $\underline{b_2} > \widetilde{b}_t\left(b_{t-1}^*(h_{t-1}, 1), 2\right)$. Then, $\widetilde{b}_t\left(\underline{b_2}, 1\right) > b_{t-1}^*(h_{t-1}, 1) > \underline{b_1}$.

By the structure of the process, the probability that the history h_τ contains state 1 and state 2 while τ goes to infinity is equal to one. ■

These results hold for any concave utility function. This comes from the fact that transfers in each state must belong to a closed interval. Consequently, if the first best contract transition given by $\widetilde{b}_t\left(b_{t-1}^*(h_{t-1}, s), z\right)$ belongs to the interval, then there is a first-best self-enforcing contract. By definition, any first-best contract is stationary since transfers do not depend on the history but only on the actual state of nature. For any no first-best self-enforcing contract, boundaries constrain the value of transfers. In the two state case, the non-trivial self-enforcing contract (NTSEC) converges monotonically to a stationary contract where the transfer is upper bounded in state 1 or lower bounded in state 2.

In the case where the number of states of nature is higher than 2, the NTSEC does not converge to a stationary contract. The reason is transfers in intermediate states of nature (state $2, 3, ..., S-1$), it could be optimal to have history-dependent transfers. For example, in the 3-state case, transfer in state 2 could take different values depending of the history. But, if I define partial history-dependent stationarity, which says that transfers in any state depend only of the part of the history in which state 1 and S was realized, I can obtain a lemma similar to Lemma 2 using partial history-dependent stationarity for any number of states of nature.

Now, I am able to study the existence of a NTSEC. To prove the existence of such contract, I can only look for the existence of a stationary contract which satisfies the self-enforcing constraints. By Lemma 2, if there is a NTSEC δ^*, then this contract converges monotonically to a stationary contract δ'. The contract δ' which must be self-enforcing since a self-enforcing contract must be self-enforcing in any state of nature and at any period. Consequently, looking for the existence of a stationary self-enforcing contract is enough to prove the existence of a NTSEC.

Proposition 1 *Let ρ be the probability of being in the state of nature 1 and $y_2 > y_1$. If $e^{r(y_2-y_1)} \geq \left[1 + \frac{1-\beta}{\beta^2 \rho * (1-\rho)}\right]$, then there are some values of $\overline{V}$ for which the solution to (4) is not the TSEC.*

Proof. By Lemma 2, each optimal contract δ^* converges to a stable contract δ'. Then, if δ' is not self-enforcing, neither is δ^*.

Take δ' and assume that this contract gives at any period b_1' if the state of nature is 1 and b_2' otherwise. Let U' and V' be the gain in utility of agent 1 and 2 respectively with the contract δ'. Suppose that δ' is self-enforcing. Then,

$$\begin{aligned}
-e^{-r(y_1-b_1')} + e^{-ry_1} + \beta E_s[U'] &\geq 0 \\
-e^{-r(y_2-b_2')} + e^{-ry_2} + \beta E_s[U'] &\geq 0 \\
-e^{-r(\overline{w}+b_1')} + e^{-r\overline{w}} + \beta E_s[V'] &\geq 0 \\
-e^{-r(\overline{w}+b_2')} + e^{-r\overline{w}} + \beta E_s[V'] &\geq 0
\end{aligned}$$

I have supposed that $y_2 > y_1$. This means that agent 1 is relatively more rich in state 2 than in state 1. Then, the optimal transfer must be negative in state 1 and positive in state 2.

If I take a look at the participation constraints, I see that only two constraints are really constraining.

$$\begin{aligned}
-e^{-r(y_2-b_2')} + e^{-ry_2} + \beta E_s[U'] &\geq 0 \\
-e^{-r(\overline{w}+b_1')} + e^{-r\overline{w}} + \beta E_s[V'] &\geq 0
\end{aligned}$$

The other two are not because in those cases, the agent receives some amount. Then, they do not want to break the contract. By definition, U' and V' are stable. I can compute their value by using the Bellman equation.

$$\begin{aligned}
U' &= \rho\left(-e^{-r(y_1-b_1')} + e^{-ry_1}\right) + (1-\rho)\left(-e^{-r(y_2-b_2')} + e^{-ry_2}\right) + \beta U' \\
U' &= \frac{1}{1-\beta}\left[\rho\left(-e^{-r(y_1-b_1')} + e^{-ry_1}\right) + (1-\rho)\left(-e^{-r(y_2-b_2')} + e^{-ry_2}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
V' &= \left[\rho\left(-e^{-rb_1'} + 1\right) + (1-\rho)\left(-e^{-rb_2'} + 1\right)\right] + \beta V' \\
V' &= \frac{1}{1-\beta}\left[\rho\left(-e^{-rb_1'} + 1\right) + (1-\rho)\left(-e^{-rb_2'} + 1\right)\right]
\end{aligned}$$

I replace U' and V' in the preceding constraints. Now, I must isolate b_2' in the first constraint.

$$\begin{aligned}
-e^{-r(y_2-b_2')}+e^{-ry_2}+\frac{\beta}{1-\beta}\left[\rho\left(-e^{-r(y_1-b_1')}+e^{-ry_1}\right)+(1-\rho)\left(-e^{-r(y_2-b_2')}+e^{-ry_2}\right)\right] &\geq 0\\
\frac{\beta\rho}{1-\beta}\left(-e^{-r(y_1-b_1')}+e^{-ry_1}\right)+\frac{1-\beta\rho}{1-\beta}\left(-e^{-r(y_2-b_2')}+e^{-ry_2}\right) &\geq 0\\
\beta\rho\left(-e^{-r(y_1-b_1')}+e^{-ry_1}\right)+(1-\beta\rho)\left(-e^{-r(y_2-b_2')}+e^{-ry_2}\right) &\geq 0\\
\beta\rho\left(-e^{r(y_2-y_1)}e^{rb_1'}+e^{r(y_2-y_1)}\right)+(1-\beta\rho)\left(-e^{rb_2'}+1\right) &\geq 0
\end{aligned}$$

And I obtain :

$$\frac{\beta\rho}{1-\beta\rho}\left(-e^{r(y_2-y_1)}e^{rb_1'}+e^{r(y_2-y_1)}\right)+1 \geq e^{rb_2'}$$

Graphically, this condition is represented by Figure 1.

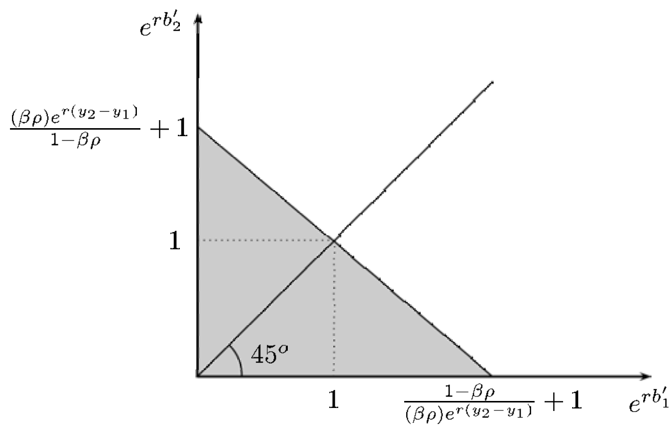

Figure 1: First Constraint

I can proceed in the same way with the second constraint.

$$\begin{aligned}
-e^{-rb_1'}+1+\beta\left[\frac{\rho}{1-\beta}\left(-e^{-rb_1'}+1\right)+\frac{(1-\rho)}{1-\beta}\left(-e^{-rb_2'}+1\right)\right] &\geq 0\\
-e^{-rb_1'}+\beta\left[\frac{\rho}{1-\beta}\left(-e^{-rb_1'}\right)+\frac{(1-\rho)}{1-\beta}\left(-e^{-rb_2'}\right)\right] &\geq -\frac{1}{1-\beta}\\
-e^{-rb_1'}+\beta\left[\frac{\rho}{1-\beta}\left(-e^{-rb_1'}\right)+\frac{(1-\rho)}{1-\beta}\left(-e^{-rb_2'}\right)\right] &\geq -\frac{1}{1-\beta}\\
(1-\beta+\beta\rho)\left(-e^{-rb_1'}\right)+(\beta-\beta\rho)\left(-e^{-rb_2'}\right) &\geq -1
\end{aligned}$$

And I obtain :

$$\frac{1-\beta+\beta\rho}{\beta-\beta\rho}\left(-e^{-rb_1'}\right)+\frac{1}{\beta-\beta\rho} \geq e^{-rb_2'}$$
$$\frac{\beta-\beta\rho}{1-(1-\beta+\beta\rho)\left(e^{-rb_1'}\right)} \leq e^{rb_2'}$$

Now, I can graph this condition (See Figure 2)

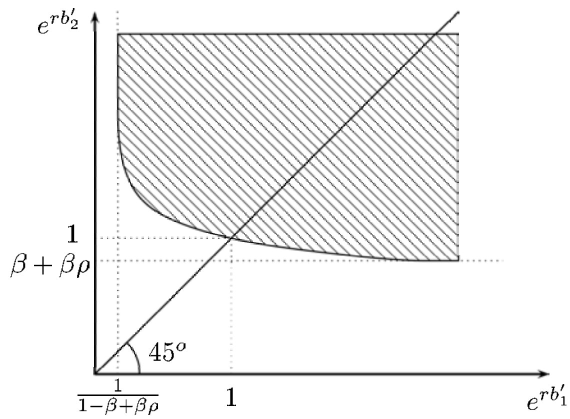

Figure 2: Second constraint

I know that the frontier must have the point $(1,1)$ since the TSEC is self-enforcing. If I combine the two constraints, I obtain Figure 3.

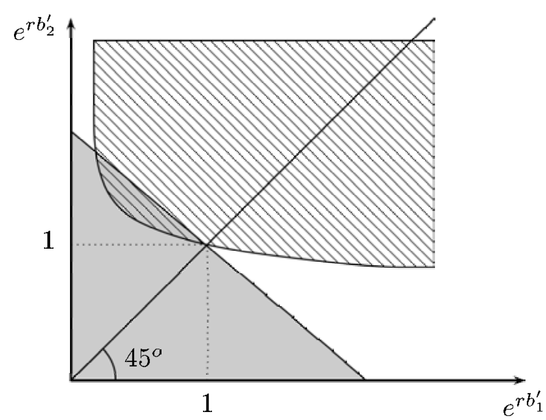

Figure 3: Both Constraints

The grey and hatched region is the set of all contracts like δ'. To know if there exists such contracts, I must analyze the slope of the two constraints at the point $(1,1)$. Lets begin with the first constraint.

$$\frac{d(e^{rb_2'})}{d(e^{rb_1'})} = -\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}$$

For the second one, I obtain:

$$\begin{aligned}\frac{d(e^{rb_2'})}{d(e^{rb_1'})} &= \frac{d}{d(e^{rb_1'})}\left(\frac{\beta-\beta\rho}{1-(1-\beta+\beta\rho)\,e^{-rb_1'}}\right)\\ \frac{d(e^{rb_2'})}{d(e^{rb_1'})} &= -\frac{\beta-\beta\rho}{\left[1-(1-\beta+\beta\rho)\,e^{-rb_1'}\right]^2}(1-\beta+\beta\rho)\,e^{-rb_1'}\end{aligned}$$

If I evaluate this slope at $(1,1)$, I obtain

$$\frac{d(e^{rb_2'})}{d(e^{rb_1'})} = -\frac{(1-\beta+\beta\rho)}{\beta-\beta\rho}$$

In order for self-enforcing contracts other than the TSEC to exist, the slope of the second constraint must be larger than the slope of the first constraint.

$$\begin{aligned}-\frac{(1-\beta+\beta\rho)}{\beta-\beta\rho} &\geq -\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\\ \frac{(1-\beta\rho)(1-\beta+\beta\rho)}{\beta\rho(\beta-\beta\rho)} &\leq e^{r(y_2-y_1)}\\ 1+\frac{1-\beta}{\beta\rho(\beta-\beta\rho)} &\leq e^{r(y_2-y_1)}\end{aligned}$$

Then, the slope of the first constraint is lower than the slope of the second if $e^{r(y_2-y_1)} > 1+\frac{1-\beta}{\beta\rho*(\beta-\beta\rho)}$. ■

For the moment, I do not know if the optimal contract is first-best. Proposition 1 tells us only under which conditions a non-trivial solution to (4) exists. Proposition 2 gives the condition to have a self-enforcing first-best contract.

Proposition 2 *Let ρ be the probability of being in the state of nature 1 and $y_2 > y_1$. If $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta^2\rho*(1-\rho)}\right]^2$, then there is some value of $\overline{V}$ such that the optimal contract is first-best.*

Proof. Suppose that the optimal first-best contract is b_1^{fb}, b_2^{fb} and let U^{fb} and V^{fb} be the gain in utility for agent 1 and 2 with the contract δ^{fb}. The first-best contract is self-enforcing if it fulfills the self-enforcing constraints. In the proof of Proposition (1), I state that only two self-enforcing constraints are relevant.

$$\begin{aligned}-e^{-r(y_2-b_2^{fb})}+e^{-ry_2}+\beta E_s\left[U^{fb}\right] &\geq 0\\ -e^{-r(\overline{w}+b_1^{fb})}+e^{-r\overline{w}}+\beta E_s\left[V^{fb}\right] &\geq 0\end{aligned}$$

By (5), I know that the first-best contract is given by the following relation:

$$e^{rb_2^{fb}} = e^{\frac{r}{2}(y_2-y_1)}e^{rb_1^{fb}}$$

Let A be the NTSEC that fulfills both self-enforcing constraints with equality. Then, some first-best contracts are self-enforcing if A is on the left side of the first-best contract line. To proceed, I must find the solutions to the equations for the constraints. Since the TSEC satisfies the constraints, I must focus on the other solution (point A). Let $\left(b_1^A, b_2^A\right)$ be the values of the transfers at point A and let U^A and V^A be the gain in utility of agent 1 and 2 with the contract δ^A. Then, point A represents the non-trivial solution of

$$\begin{aligned} -e^{-r\left(y_2-b_2^A\right)}+e^{-ry_2}+\beta E_s\left[U^A\right] &= 0 \\ -e^{-r\left(\overline{w}+b_1^A\right)}+e^{-r\overline{w}}+\beta E_s\left[V^A\right] &= 0 \end{aligned}$$

In the previous proof, I have found that those equations can be written as:

$$\begin{aligned} \frac{\beta\rho}{1-\beta\rho}\left(-e^{r(y_2-y_1)}e^{rb_1^A}+e^{r(y_2-y_1)}\right)+1 &= e^{rb_2^A} \\ \frac{\beta-\beta\rho}{1-(1-\beta+\beta\rho)\left(e^{-rb_1^A}\right)} &= e^{rb_2^A} \end{aligned}$$

By solving this system of equations, I find that the non-trivial solution is :

$$\begin{aligned} e^{rb_2^A} &= \left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)(\beta-\beta\rho)+(\beta-\beta\rho) \\ e^{rb_1^A} &= 1+\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}-(\beta-\beta\rho)-\left(\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\right)(\beta-\beta\rho) \end{aligned}$$

If I calculate the slope of the line which connects point A to the origin, $\frac{e^{rb_1^A}}{e^{rb_2^A}}$, I find:

$$\begin{aligned} \frac{e^{rb_1^A}}{e^{rb_2^A}} &= \frac{1+\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}-(\beta-\beta\rho)-\left(\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\right)(\beta-\beta\rho)}{\left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)(\beta-\beta\rho)+(\beta-\beta\rho)} \\ &= \frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\frac{\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}+1-\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}(\beta-\beta\rho)-(\beta-\beta\rho)}{\left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)(\beta-\beta\rho)+(\beta-\beta\rho)} \\ &= \frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\left(\frac{\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}+1}{\left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)(\beta-\beta\rho)+(\beta-\beta\rho)}-1\right) \\ &= \frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\left(\frac{1}{\beta-\beta\rho}-1\right) \end{aligned}$$

Now, I must compare this result with the slope of the line of first-best contracts. If the slope of the first-best contract line is lower than the slope I find above, then some first-best contracts are self-enforcing.

$$\begin{aligned}
\frac{e^{rb_1^{fb}}}{e^{rb_2^{fb}}} &\geq \frac{e^{rb_1^A}}{e^{rb_2^A}} \\
\frac{1}{e^{\frac{r}{2}(y_2-y_1)}} &\geq \frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\left(\frac{1}{\beta-\beta\rho}-1\right) \\
e^{\frac{r}{2}(y_2-y_1)} &\geq \frac{1-\beta\rho}{\beta\rho}\left(\frac{1-\beta+\beta\rho}{\beta-\beta\rho}\right) \\
e^{\frac{r}{2}(y_2-y_1)} &\geq \frac{1-\beta+\beta\rho-\beta\rho+\beta^2\rho-\beta^2\rho^2}{\beta\rho\,(\beta-\beta\rho)} \\
e^{\frac{r}{2}(y_2-y_1)} &\geq 1+\frac{1-\beta}{\beta\rho\,(\beta-\beta\rho)}
\end{aligned}$$

Then, if $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta\rho(\beta-\beta\rho)}\right]^2$, there exist some values of $\overline{V}$ such that the optimal contract is first-best. ■

The idea of the proof is the following: the first-best relation given by (5) must be compared with the non-trivial contract solving the two self-enforcing constraints. Precisely, I must compare ratios $\frac{e^{rb_1^{fb}}}{e^{rb_2^{fb}}}$ and $\frac{e^{rb_1^A}}{e^{rb_2^A}}$ where b_s^{fb} is the transfer in state s under a first best contract [8] and b_s^A is the transfer in state s when the contract is the non-trivial one solving self-enforcing constraints. Figure 4 illustrates the idea.

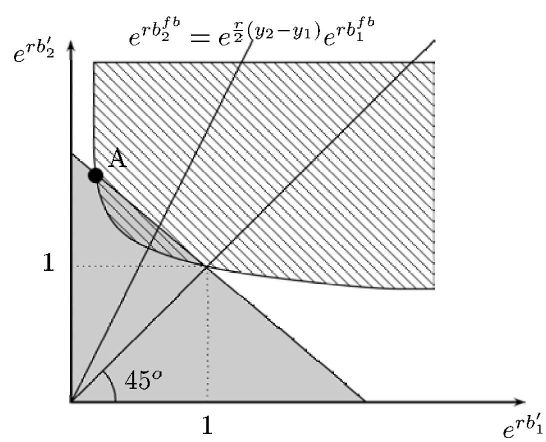

Figure 4: First-Best Contracts and Constraints

From the two preceding propositions, if $y_2 - y_1$ increases, then the optimal contract will become non-trivial once $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta\rho(\beta-\beta\rho)}\right]$ and when $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta\rho(\beta-\beta\rho)}\right]^2$, then the optimal contract will become first best. Those results can be viewed as the dual solution from Proposition 4 of Thomas and Worrall [19] which says that there is a discount factor β^* such that, for all $\beta > \beta^*$, some optimal contracts are first-best and there is a $\beta_* < \beta^*$ such that for all

[8] Mathematically, I find that the ratio $\frac{e^{rb_1^{fb}}}{e^{rb_2^{fb}}}$ is constant for any first-best contract.

$\beta \in [\beta_*, \beta^*)$, the optimal contract is non-trivial but not first-best.[9]

I prove Proposition 2 by finding the condition such that a first-best contract satisfies all self-enforcing constraints. But, what can I say about the optimal contract? Kocherlakota [12] proves that, when some optimal contracts are first-best, then the expected utility converges to a utility level given by a self-enforcing first-best contract.[10] I can rewrite this proposition in a equivalent way in term of contracts

Proposition 3 *Suppose that some first-best contract is optimal. Then, all optimal contracts converge to a first-best contract.*

Proof. See Proposition 4.1 of Kocherlakota. ■

I say that a contract δ is first-best convergent if it converges to a first-best contract. This definition will be very useful in Section 5.

3.2 Pareto Frontier

In the previous section, I derive the condition to have a NTSEC. Here, I want to show how the self-enforcing constraints affect the optimality of the contract. To do so, I use the Pareto frontier in either case where a first-best contract is or is not self-enforcing and I compare with the Pareto frontier when there is no self-enforcing constraint. I first begin with the Pareto frontier when there is no self-enforcing constraint.

Proposition 4 *Without self-enforcing constraints, the Pareto frontier is given by:*

$$U^U\left(\overline{V}\right) = \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right] - \frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}\right]$$

with $\overline{V} \in \left[0, \frac{1}{(1-\beta)e^{r\overline{w}}}\left(1 - \frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{E_s\left[e^{-ry_s}\right]}\right)\right]$.

Proof. First, with the assumption of constant revenues for agent 2, I can rewrite the participation constraint. By Proposition 4.1 of Kocherlakota [12], I have that $V(\delta, h_t) = \overline{V}$. Then,

$$\begin{aligned} E_s\left[-e^{-r(\overline{w}_s+b_s^{fb})} + e^{-r\overline{w}_s} + \beta\overline{V}\right] &\geq \overline{V} \\ E_s\left[-e^{-rb_s^{fb}} + 1\right] &\geq (1-\beta)\,e^{r\overline{w}}\overline{V} \\ E_s\left[e^{-rb_s^{fb}}\right] &= 1-(1-\beta)\,e^{r\overline{w}}\overline{V} \end{aligned}$$

I know from (5) that the relation between transfers is given by:

$$b_2^{fb} = b_1^{fb} + \frac{1}{2}(y_2 - y_1)$$

[9] It is possible to write conditions to have a NTSEC or a first-best self-enforcing contract with beta on the left side but conditions become a bit messy

[10] See Proposition 4.1 of Kocherlakota [12].

By introducing this result into the participation constraint for agent 2, I obtain:

$$\begin{aligned}
\rho\left(e^{-rb_1^{fb}}\right)+(1-\rho)\left(e^{-rb_2^{fb}}\right) &= 1-(1-\beta)\,e^{r\overline{w}}\overline{V} \\
\rho\left(e^{-rb_1^{fb}}\right)+(1-\rho)\left(e^{-r\left(b_1^{fb}+\frac{1}{2}(y_2-y_1)\right)}\right) &= 1-(1-\beta)\,e^{r\overline{w}}\overline{V} \\
e^{-rb_1^{fb}} &= \frac{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}{\rho+(1-\rho)\left(e^{\frac{r}{2}(y_1-y_2)}\right)}
\end{aligned}$$

And $e^{-rb_2^{fb}}$ is given by:

$$e^{-rb_2^{fb}}=\frac{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}{\rho\left(e^{\frac{r}{2}(y_2-y_1)}\right)+(1-\rho)}$$

Then, I am able to define the Pareto frontier explicitly by introducing b_1^{fb} and b_2^{fb} in the utility function of agent 1.

$$\begin{aligned}
U^{fb} &= E_s\left[-e^{-r(y_s-b_s^{fb})}+e^{-ry_s}+\beta U^{fb}\right] \\
U^{fb} &= \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right]-\rho e^{-r(y_1-b_1^{fb})}-(1-\rho)\,e^{-r(y_2-b_2^{fb})}\right] \\
U^{fb} &= \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right]-\frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}\right]
\end{aligned}$$

The maximum value for $\overline{V}$ is reached when $U^{fb}=0$.

$$\begin{aligned}
0 &= \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right]-\frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}_{MAX}}\right] \\
E_s\left[e^{-ry_s}\right] &= \frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}_{MAX}} \\
\overline{V}_{MAX} &= \frac{1}{(1-\beta)e^{r\overline{w}}}\left(1-\frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{E_s\left[e^{-ry_s}\right]}\right)
\end{aligned}$$

■

Figure 5 represents the unconstrained Pareto frontier when there are no self-enforcing constraints.

Without self-enforcing constraints, this Pareto frontier is attainable everywhere. This is not the case when I add self-enforcing constraints. With self-enforcing constraints, as shown above, there are two possibilities: either some first-best contracts are self-enforcing or no first-best contract is. In the following proposition, I present the Pareto frontier if there is no self-enforcing first-best contracts.

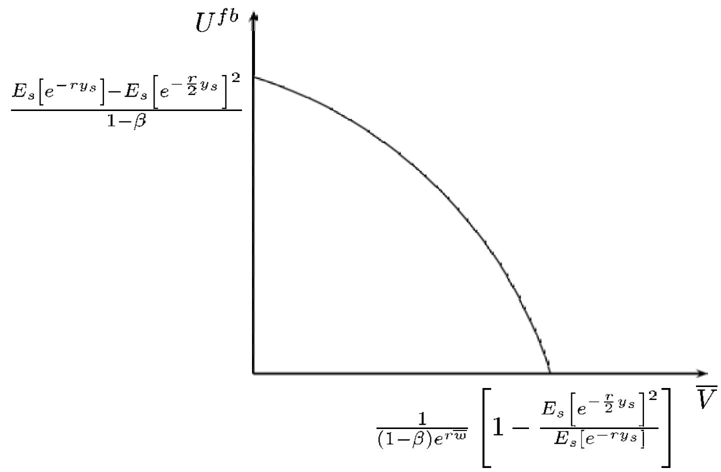

Figure 5: Unconstrained Pareto Frontier

Proposition 5 *Suppose that* $\left[1+\frac{1-\beta}{\beta^2\rho*(1-\rho)}\right]^2 > e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta^2\rho*(1-\rho)}\right]$ *and let*

$$
\begin{aligned}
e^{rb_1^A} &= 1-\beta+\beta\rho+\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\left(1-\beta+\beta\rho\right) \\
e^{rb_2^A} &= \left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)(\beta-\beta\rho)+(\beta-\beta\rho) \\
V^A &= \frac{(1-\rho)\,e^{-r\overline{w}}}{1-\beta+\beta\rho}\left(1-e^{-rb_2^A}\right) \\
V^{MAX} &= \frac{(1-\rho)\,e^{-r\overline{w}}}{(1-\beta\rho)\,(1-\beta+\beta\rho)}\left(1-e^{-rb_2^A}\right)
\end{aligned}
$$

Then,

- *if* $\overline{V}\in\left[0,V^A\right]$, *then the optimal contract is given by:*
  - $b_t\left(h_t,s\right)=b_1^A$ *if the state of nature s is 1.*
  - $b_t\left(h_t,s\right)=\frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\beta]e^{r\overline{w}}\overline{V}}$ *if the history is* $h_t=(2,2,...,2)$.
  - $b_t\left(h_t,s\right)=b_2^A$ *otherwise.*
- *if* $\overline{V}\in\left[V^A,V^{MAX}\right]$, *then the optimal contract is given by:*
  - $b_t\left(h_t,s\right)=b_2^A$ *if the state of nature s is 2.*
  - $b_t\left(h_t,s\right)=\frac{\rho(1-\beta+\beta\rho)}{\rho(1-\beta+\beta\rho)+(1-\rho)\left(1-e^{-rb_2^A}\right)-(1-\rho\beta)(1-\beta+\beta\rho)*e^{rd}*\overline{V}}$ *if the history is* $h_t=(1,1,...,1)$.
  - $b_t\left(h_t,s\right)=b_1^A$ *otherwise.*

And

- *if* $\overline{V} \in [0, V^A]$, *then the Pareto frontier is given by:*

$$U^{SE}(\overline{V}) = \frac{\rho\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}\right)}{(1-\beta+\beta\rho)(1-\rho\beta)} + \frac{(1-\rho)*e^{-ry_2}}{(1-\beta+\beta\rho)}\left(1-\frac{1-\rho}{1-\rho-e^{r\overline{w}}*(1-\beta+\beta\rho)*\overline{V}}\right)$$

- *if* $\overline{V} \in [V^A, V^{MAX}]$, *then the Pareto frontier is given by:*

$$U^{SE}(\overline{V}) = \frac{\rho e^{-ry_1}}{1-\beta\rho}\left(1-\frac{\rho(1-\beta+\beta\rho)}{\gamma}\right)$$

where $\gamma = \rho(1-\beta+\beta\rho)+(1-\rho)\left(1-e^{-rb_2^A}\right)-(1-\rho\beta)(1-\beta+\beta\rho)*e^{r\overline{w}}\overline{V}$.

Proof. By Proposition 1 and 2, I already know that there is no first-best self-enforcing contract. By Lemma 2, the optimal contract converges monotonically to the contract given by the non-trivial solution of the following 2 self-enforcing constraints:

$$\frac{\beta\rho}{1-\beta\rho}\left(-e^{r(y_2-y_1)}e^{rb_1'}+e^{r(y_2-y_1)}\right)+1 = e^{rb_2'}$$

$$\frac{\beta-\beta\rho}{1-(1-\beta+\beta\rho)\left(e^{-rb_1'}\right)} = e^{rb_2'}$$

which is

$$e^{rb_1^A} = 1-\beta+\beta\rho+\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}(1-\beta+\beta\rho)$$

$$e^{rb_2^A} = \left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)(\beta-\beta\rho)+(\beta-\beta\rho)$$

Graphically,

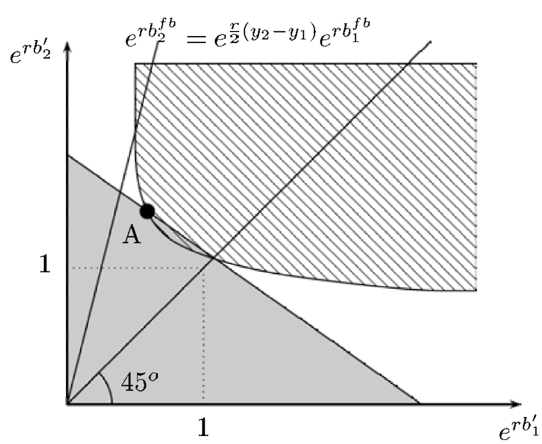

Figure 6: Stationary contract

Let V^A be the utility for agent 2 at point A. Then,

$$V^A = \rho\left(-e^{-r(\overline{w}+b_1^A)} + e^{-r\overline{w}} + \beta V^A\right) + (1-\rho)\left(-e^{-r(\overline{w}+b_2^A)} + e^{-r\overline{w}} + \beta V^A\right)$$

But, the stationary contract satisfies the relevant participation constraint with equality. Then:

$$-e^{-r(\overline{w}+b_1^A)} + e^{-r\overline{w}} + \beta V^A = 0$$

Then, I have:

$$\begin{aligned} V^A &= (1-\rho)\left(-e^{-r(\overline{w}+b_2^A)} + e^{-r\overline{w}} + \beta V^A\right) \\ V^A &= \frac{e^{-r\overline{w}}(1-\rho)\left(1-e^{-rb_2^A}\right)}{1-\beta(1-\rho)} \end{aligned}$$

Then, if $\overline{V} = V^A$, the optimal contract is the contract represented by point A. If $\overline{V} \neq V^A$, then the optimal contract is different than the contract represented by point A, but must monotonically converge to the A-contract. In the proof of Lemma 2, I have seen that a contract can only differ from a stable contract at the beginning and until the state of nature switches. In other words, the transfer in state 1 at period t can be different from b_1^A if state 2 is not yet realized in the t first periods and the transfer in state 2 at period t can be different from b_2^A if state 1 is not yet realized in the t first periods.

This results in two types of contracts:

Type 1:

- The transfer at period t is b_1^A if the state of nature is 1.
- The transfer at period t is $b_2^* \leq b_2^A$ if the state of nature is 2 at period t and the state of nature was not realized in the first $t-1$ periods.
- The transfer at period t is b_2^A if the state of nature is 2 at period t and the state of nature was realized in the first $t-1$ periods.

Type 2:

- The transfer at period t is b_2^A if the state of nature is 2.
- The transfer at period t is $b_1^* \geq b_1^A$ if the state of nature is 1 at period t and the state of nature was not realized in the first $t-1$ periods.
- The transfer at period t is b_1^A if the state of nature is 1 at period t and the state of nature was realized in the first $t-1$ periods.

The type 1 contract gives more utility to agent 1 and less to agent 2 and the opposite is true for type 2 contract. Then, when $\overline{V} \leq V^A$, the optimal contract is type 1 and when $\overline{V} \geq V^A$, the optimal contract is type 2.

Now, I must calculate the transfer in the first t periods in term of $\overline{V}$. Let's begin with the case where $\overline{V} \leq V^A$. Then,

$$\overline{V} \;\; = \rho\left(-e^{-r(\overline{w}+b_1^A)} + e^{-r\overline{w}} + \beta V^A\right) + (1-\rho)\left(-e^{-r(\overline{w}+b_2^*)} + e^{-r\overline{w}} + \beta\overline{V}\right)$$

But, I find b_1^A by using the self-enforcing constraint :

$$-e^{-r(\overline{w}+b_1^A)} + e^{-r\overline{w}} + \beta V^A = 0$$

I obtain:

$$\begin{aligned}
\overline{V} &= (1-\rho)\left(-e^{-r(\overline{w}+b_2^*)}+e^{-r\overline{w}}+\beta\overline{V}\right)\\
\overline{V} &= (1-\rho)\left(-e^{-r(\overline{w}+b_2^*)}+e^{-r\overline{w}}\right)+(1-\rho)\beta\overline{V}\\
(1-(1-\rho)\beta)\overline{V} &= (1-\rho)\left(-e^{-r(\overline{w}+b_2^*)}+e^{-r\overline{w}}\right)\\
\left(\frac{1-(1-\rho)\beta}{(1-\rho)}\right)\overline{V} &= \left(-e^{-r(\overline{w}+b_2^*)}+e^{-r\overline{w}}\right)\\
\left(\frac{1-(1-\rho)\beta}{(1-\rho)}\right)e^{r\overline{w}}\overline{V} &= \left(1-e^{-rb_2^*}\right)
\end{aligned}$$

And

$$\begin{aligned}
e^{-rb_2^*} &= 1-\frac{1-(1-\rho)\beta}{(1-\rho)}e^{r\overline{w}}\overline{V}\\
e^{rb_2^*} &= \frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\beta]\,e^{r\overline{w}}\overline{V}}
\end{aligned}$$

If $\overline{V}=0$, then $b_2^*=0$. If $\overline{V}=V^A$, then:

$$\begin{aligned}
e^{rb_2^*} &= \frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\beta]\,e^{r\overline{w}}\frac{e^{-r\overline{w}}(1-\rho)\left(1-e^{-rb_2^A}\right)}{1-\beta(1-\rho)}}\\
e^{rb_2^*} &= \frac{(1-\rho)}{(1-\rho)-(1-\rho)\left(1-e^{-rb_2^A}\right)}\\
e^{rb_2^*} &= e^{rb_2^A}
\end{aligned}$$

Now, I examine the case where $\overline{V}>V^A$.

$$\begin{aligned}
\overline{V} &= \rho\left(-e^{-r(\overline{w}+b_1^*)}+e^{-r\overline{w}}+\beta\overline{V}\right)+(1-\rho)\left(-e^{-r(\overline{w}+b_2^A)}+e^{-r\overline{w}}+\beta V^A\right)\\
(1-\beta\rho)\overline{V} &= \rho\left(-e^{-r(\overline{w}+b_1^*)}+e^{-r\overline{w}}\right)+(1-\rho)\left(-e^{-r(\overline{w}+b_2^A)}+e^{-r\overline{w}}+\beta V^A\right)\\
(1-\beta\rho)e^{r\overline{w}}\overline{V} &= \rho\left(-e^{-rb_1^*}+1\right)+(1-\rho)\left(-e^{-rb_2^A}+1+\beta e^{r\overline{w}}V^A\right)
\end{aligned}$$

Then:

$$\begin{aligned}
(1-\beta\rho)e^{r\overline{w}}\overline{V}-(1-\rho)\left(-e^{-rb_2^A}+1+\beta e^{r\overline{w}}V^A\right) &= \rho\left(-e^{-rb_1^{sb}}+1\right)\\
1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho}\left(-e^{-rb_2^A}+1+\beta e^{r\overline{w}}V^A\right) &= e^{-rb_1^{sb}}
\end{aligned}$$

Moreover, I have already found that $V^A = \frac{(1-\rho)\left(-e^{-r(\overline{w}+b_2^A)}+e^{-r\overline{w}}\right)}{1-\beta(1-\rho)}$. Hence,

$$\begin{aligned}
e^{-rb_1^*} &= 1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho}\left(-e^{-rb_2^A}+1+\beta e^{r\overline{w}}\frac{(1-\rho)\left(-e^{-r(\overline{w}+b_2^A)}+e^{-r\overline{w}}\right)}{1-\beta(1-\rho)}\right)\\
e^{-rb_1^{sb}} &= 1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho}\left(-e^{-rb_2^A}+1+\beta\frac{(1-\rho)\left(-e^{-rb_2^A}+1\right)}{1-\beta(1-\rho)}\right)\\
e^{-rb_1^{sb}} &= 1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho(1-\beta+\beta\rho)}\left(1-e^{-rb_2^A}\right)\\
e^{rb_1^{sb}} &= \frac{\rho(1-\beta+\beta\rho)}{\rho(1-\beta+\beta\rho)+(1-\rho)\left(1-e^{-rb_2^A}\right)-(1-\rho\beta)(1-\beta+\beta\rho)*e^{rd}*\overline{V}}
\end{aligned}$$

Let V^{MAX} be the maximal value for $\overline{V}$. Then, V^{MAX} is reached when $b_1^*=0$. To have $b_1^*=0$, I must have:

$$\begin{aligned}
(1-\beta\rho)(1-\beta+\beta\rho)e^{r\overline{w}}V^{MAX} &= (1-\rho)\left(1-e^{-rb_2^A}\right)\\
V^{MAX} &= \frac{(1-\rho)e^{-r\overline{w}}}{(1-\beta\rho)(1-\beta+\beta\rho)}\left(1-e^{-rb_2^A}\right)
\end{aligned}$$

The previous part of the proof gives the optimal contract relative to the value of $\overline{V}$. Then, if I replace those values in the utility function of agent 1, I obtain the Pareto frontier equation. Let's begin with the case where $\overline{V}\in\left[0,V^A\right]$. In this case, the utility function of agent 1 is given by:

$$U^{OP}\left(\overline{V}\right)=\rho\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}+\beta U^A\right)+(1-\rho)\left(-e^{-ry_2}e^{rb_2^*}+e^{-ry_2}+\beta U^{OP}\left(\overline{V}\right)\right)$$

With

$$U^A=\rho\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}+\beta U^A\right)+(1-\rho)\left(-e^{-ry_2}e^{rb_2^A}+e^{-ry_2}+\beta U^A\right)$$

Because $-e^{-r(y_2-b_2^A)}+e^{-ry_2}+\beta U^A=0$ by the self-enforcing constrain, I obtain:

$$\begin{aligned}
U^A &= \rho\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}+\beta U^A\right)\\
U^A &= \frac{\rho}{(1-\rho\beta)}\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}\right)
\end{aligned}$$

Then,

$$\begin{aligned}
U^{OP}\left(\overline{V}\right) &= \frac{\rho}{(1-\rho\beta)}\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}\right)\\
&\quad+(1-\rho)\left(-e^{-ry_2}e^{rb_2^*}+e^{-ry_2}+\beta U^{OP}\left(\overline{V}\right)\right)\\
(1-\beta+\beta\rho)U^{OP}\left(\overline{V}\right) &= \frac{\rho}{(1-\rho\beta)}\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}\right)+(1-\rho)\left(-e^{-ry_2}e^{rb_2^*}+e^{-ry_2}\right)\\
U^{OP}\left(\overline{V}\right) &= \frac{\rho}{(1-\beta+\beta\rho)(1-\rho\beta)}\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}\right)\\
&\quad+\frac{(1-\rho)}{(1-\beta+\beta\rho)}\left(-e^{-ry_2}e^{rb_2^*}+e^{-ry_2}\right)
\end{aligned}$$

If I substitute $e^{rb_2^*}=\frac{1-\rho}{1-\rho-e^{r\overline{w}}(1-\beta+\beta\rho)\overline{V}}$, I obtain:

$$U^{OP}\left(\overline{V}\right)=\frac{\rho\left(-e^{-r\left(y_1-b_1^A\right)}+e^{-ry_1}\right)}{\left(1-\beta+\beta\rho\right)\left(1-\rho\beta\right)}+\frac{\left(1-\rho\right)*e^{-ry_2}}{\left(1-\beta+\beta\rho\right)}\left(1-\frac{1-\rho}{1-\rho-e^{r\overline{w}}\left(1-\beta+\beta\rho\right)\overline{\overline{V}}}\right)$$

$$U^{OP}\left(\overline{V}\right)=\frac{\rho\left(-e^{-r\left(y_1-b_1^A\right)}+e^{-ry_1}\right)}{\left(1-\beta+\beta\rho\right)\left(1-\rho\beta\right)}+\frac{\left(1-\rho\right)*e^{-ry_2}}{\left(1-\beta+\beta\rho\right)}\left(\frac{-e^{r\overline{w}}*\left(1-\beta+\beta\rho\right)*\overline{V}}{1-\rho-e^{r\overline{w}}\left(1-\beta+\beta\rho\right)\overline{\overline{V}}}\right)$$

Now, for the case where $\overline{V}\in\left[V^A,V^{MAX}\right]$. In this case, the utility function for agent 1 is given by:

$$U^{OP}\left(\overline{V}\right)=\rho\left(-e^{-r\left(y_1-b_1^*\right)}+e^{-ry_1}+\beta U^{OP}\left(\overline{V}\right)\right)+\left(1-\rho\right)\left(-e^{-r\left(y_2-b_2^A\right)}+e^{-ry_2}+\beta f\left(V^A\right)\right)$$

I already know that $-e^{-r\left(y_2-b_2^A\right)}+e^{-ry_2}+\beta U^A=0$. Then,

$$\begin{aligned}U^{OP}\left(\overline{V}\right) &= \frac{\rho}{1-\beta\rho}\left(-e^{-r\left(y_1-b_1^*\right)}+e^{-ry_1}\right)\\ U^{OP}\left(\overline{V}\right) &= \frac{\rho}{1-\beta\rho}e^{-ry_1}\\ &\left(1-\frac{\rho\left(1-\beta+\beta\rho\right)}{\rho\left(1-\beta+\beta\rho\right)+\left(1-\rho\right)\left(1-e^{-rb_2^A}\right)-\left(1-\rho\beta\right)\left(1-\beta+\beta\rho\right)e^{r\overline{w}}\overline{\overline{V}}}\right)\end{aligned}$$

■

Since the unconstrained Pareto frontier represents the maximum agent's utilities under all first-best contracts, then the Pareto frontier when no first-best contract is self-enforcing is strictly lower. Another important point to underline is the discontinuity of the Pareto frontier. Kocherlakota [12] says that the Pareto frontier is differentiable everywhere. In fact, as corrected by Koeppl [13], the Pareto frontier is not differentiable everywhere (Proposition 3.1). If I examine the Pareto frontier where a non-trivial solution exists, I find that the Pareto frontier is continuous but not differentiable everywhere[11]. The problem of differentiability occurs at the intersection of 2 segments.

This problem of discontinuity occurs also when some first-best contracts are self-enforcing. The next proposition shows the Pareto frontier in this case.

[11] The continuity is quite obvious because each segment is continuous and at intersection of two segments, the contract is defined evenly on both segments.

Proposition 6 *Suppose that* $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta^2\rho*(1-\rho)}\right]^2$ *and let*

$$
\begin{aligned}
e^{rb_1^B} &= 1-\beta+\beta\rho+(\beta-\beta\rho)\,e^{\frac{r}{2}(y_1-y_2)} \\
e^{rb_1^B} &= \beta-\beta\rho+(1-\beta+\beta\rho)\,e^{\frac{r}{2}(y_2-y_1)} \\
V^B &= \frac{(\beta-\beta\rho)\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)}{e^{r\overline{w}}\left(1-(\beta-\beta\rho)\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)\right)} \\
e^{rb_1^C} &= \frac{\beta\rho e^{-ry_1+(1-\beta\rho)e^{-ry_2}}}{\beta\rho e^{-ry_1}+(1-\beta\rho)\,e^{\frac{-r}{2}(y_1+y_2)}} \\
e^{rb_2^C} &= \frac{\beta\rho e^{-ry_1+(1-\beta\rho)e^{-ry_2}}}{\beta\rho e^{\frac{-r}{2}(y_1+y_2)}+(1-\beta\rho)\,e^{-ry_2}} \\
V^C &= \frac{\rho\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)\left(\beta\,(1-\rho)\,e^{-ry_2}-(1-\beta\rho)\,e^{\frac{r}{2}(y_1+y_2)}\right)}{e^{r\overline{w}}\,(1-\beta)\,(\beta\rho e^{ry_2}+(1-\beta\rho)\,e^{ry_1})} \\
V^{MAX} &= \frac{(1-\rho)\,e^{-r\overline{w}}}{(1-\beta\rho)\,(1-\beta+\beta\rho)}\left(1-e^{-rb_2^C}\right)
\end{aligned}
$$

Then,

- *if* $\overline{V} \in \left[0, V^B\right]$, *the optimal contract is given by:*

  - $b_t(h_t, s) = b_1^B$ *if the state of nature s is 1.*
  - $b_t(h_t, s) = \frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\beta]e^{r\overline{w}}\overline{V}}$ *if the history* $h_t = (2, 2, ..., 2)$.
  - $b_t(h_t, s) = b_2^B$ *otherwise.*

- *if* $\overline{V} \in \left[V^B, V^C\right]$, *the optimal contract is given by:*

  - $b_t(h_t, 1) = \frac{\rho+(1-\rho)e^{\frac{r}{2}(y_1-y_2)}}{1-(1-\beta)e^{r\overline{w}}\overline{V}}$.
  - $b_t(h_t, 2) = \frac{\rho e^{\frac{r}{2}(y_2-y_1)}+(1-\rho)}{1-(1-\beta)e^{r\overline{w}}\overline{V}}$.

- *if* $\overline{V} \in \left[V^C, V^{MAX}\right]$, *the optimal contract is given by:*

  - $b_t(h_t, s) = b_2^C$ *if the state of nature s is 2.*
  - $b_t(h_t, s) = \frac{\rho(1-\beta+\beta\rho)}{\rho(1-\beta+\beta\rho)+(1-\rho)\left(1-e^{-rb_2^C}\right)-(1-\rho\beta)(1-\beta+\beta\rho)*e^{r\overline{w}}*\overline{V}}$ *if the history* $h_t = (1, 1, ..., 1)$.
  - $b_t(h_t, s) = b_1^C$ *otherwise.*

And

- *if* $\overline{V} \in \left[0, V^B\right]$, *then the Pareto frontier is given by:*

$$
\begin{aligned}
U^{OP}\left(\overline{V}\right) &= \frac{(1-\rho)\,e^{-ry_2}}{1-\beta+\rho}\left(e^{rb_2^B}-\frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\,\beta]\,e^{r\overline{w}}\overline{V}}\right) \\
&\quad+\frac{\left(\rho\left(-e^{-r(y_1-b_1^B)}+e^{-ry_1}\right)+(1-\rho)\left(-e^{-ry_2}e^{rb_2^B}+e^{-ry_2}\right)\right)}{1-\beta}
\end{aligned}
$$

- *if* $\overline{V} \in \left[V^B, V^C\right]$, *then the Pareto frontier is given by:*

$$U^{sb} = \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right] - \frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}\right]$$

- *if* $\overline{V} \in \left[V^C, V^{MAX}\right]$, *then the Pareto frontier is given by:*

$$U^{SE}\left(\overline{V}\right) = \frac{\rho e^{-ry_1}}{1-\beta\rho}\left(1 - \frac{\rho\left(1-\beta+\beta\rho\right)}{\gamma}\right)$$

where $\gamma = \rho\left(1-\beta+\beta\rho\right) + (1-\rho)\left(1-e^{-rb_2^C}\right) - (1-\rho\beta)\left(1-\beta+\beta\rho\right) * e^{r\overline{w}} * \overline{V}$.

Proof. Since $e^{r(y_2-y_1)} \geq \left[1 + \frac{1-\beta}{\beta^2\rho*(1-\rho)}\right]^2$, then there exists some $\overline{V}$'s such that the optimal contract is first-best. Let's find the set of those $\overline{V}$'s.

The first step is defining the transfer in a first-best contract in terms of $\overline{V}$.

$$\begin{aligned}
\overline{V} &= \rho\left(-e^{-r(\overline{w}+b_1^{fb})} + e^{-r\overline{w}} + \beta\overline{V}\right) + (1-\rho)\left(-e^{-r(\overline{w}+b_2^{fb})} + e^{-r\overline{w}} + \beta\overline{V}\right) \\
(1-\beta)\,e^{r\overline{w}}\overline{V} &= \rho\left(1-e^{-rb_1^{fb}}\right) + (1-\rho)\left(1-e^{-rb_2^{fb}}\right) \\
1-(1-\beta)\,e^{r\overline{w}}\overline{V} &= \rho e^{-rb_1^{fb}} + (1-\rho)\,e^{-rb_2^{fb}}
\end{aligned}$$

By (5), $e^{-rb_2^{fb}} = e^{-rb_1^{fb}}e^{-\frac{r}{2}(y_2-y_1)}$. Consequently

$$\begin{aligned}
1-(1-\beta)\,e^{r\overline{w}}\overline{V} &= \rho e^{-rb_1^{fb}} + (1-\rho)\,e^{-rb_1^{fb}}e^{-\frac{r}{2}(y_2-y_1)} \\
\frac{\rho + (1-\rho)\,e^{\frac{r}{2}(y_1-y_2)}}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}} &= e^{rb_1^{fb}}
\end{aligned}$$

And

$$e^{rb_2^{fb}} = \frac{\rho e^{\frac{r}{2}(y_2-y_1)} + (1-\rho)}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}$$

Let V^B be the minimal utility of agent 2 when the contract is first best and self-enforcing. This contract is the first-best contract satisfying the self-enforcing constraint of agent 2. Let b_1^B and b_2^B be the transfers of the first-best contract for V^B. Then, I find that

$$\begin{aligned}
e^{rb_1^B} &= 1-\beta+\beta\rho+(\beta-\beta\rho)\,e^{\frac{r}{2}(y_1-y_2)} \\
e^{rb_1^B} &= \beta-\beta\rho+(1-\beta+\beta\rho)\,e^{\frac{r}{2}(y_2-y_1)} \\
V^B &= \frac{(\beta-\beta\rho)\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)}{e^{r\overline{w}}\left(1-(\beta-\beta\rho)\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)\right)}
\end{aligned}$$

Now, for the maximal $\overline{V}$, denoted V^C, given an optimal first-best contract, I must use the self-enforcing constraint of agent 1. Let b_1^C and b_2^C be the transfers of the first-best contract for

V^C which is given by

$$e^{rb_1^C} = \frac{\beta\rho e^{-ry_1+(1-\beta\rho)e^{-ry_2}}}{\beta\rho e^{-ry_1}+(1-\beta\rho)e^{\frac{-r}{2}(y_1+y_2)}}$$
$$e^{rb_2^C} = \frac{\beta\rho e^{-ry_1+(1-\beta\rho)e^{-ry_2}}}{\beta\rho e^{\frac{-r}{2}(y_1+y_2)}+(1-\beta\rho)e^{-ry_2}}$$
$$V^C = \frac{\rho\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)\left(\beta(1-\rho)e^{-ry_2}-(1-\beta\rho)e^{\frac{r}{2}(y_1+y_2)}\right)}{e^{r\overline{w}}(1-\beta)\left(\beta\rho e^{ry_2}+(1-\beta\rho)e^{ry_1}\right)}$$

If $\overline{V}\in\left[V^B,V^C\right]$, then the optimal contract is given by the first-best contract given by:

$$e^{rb_1^{fb}} = \frac{\rho+(1-\rho)e^{\frac{r}{2}(y_1-y_2)}}{1-(1-\beta)e^{r\overline{w}}\overline{V}}$$
$$e^{rb_2^{fb}} = \frac{\rho e^{\frac{r}{2}(y_2-y_1)}+(1-\rho)}{1-(1-\beta)e^{r\overline{w}}\overline{V}}$$

Now, I study the case when $\overline{V}<V^B$. Equivalent to the proof of Proposition 5, the optimal contract in this case is given by:

- The transfer at period t is b_1^B if the state of nature is 1.
- The transfer at period t is $b_2^*\leq b_2^B$ if the state of nature is 2 at period t and the other possible state of nature has not been realized at any moment during the first $t-1$ periods.
- The transfer at period t is b_2^B if the state of nature is 2 at period t and the other possible state of nature was realized at some point during the first $t-1$ periods.

To find b_2^*, I must solve

$$\overline{V} = \rho\left(-e^{-r(\overline{w}+b_1^B)}+e^{-r\overline{w}}+\beta V^B\right)+(1-\rho)\left(-e^{-r(\overline{w}+b_2^*)}+e^{-r\overline{w}}+\beta\overline{V}\right)$$

But, I have found that b_1^B by using the self-enforcing constraint :

$$-e^{-r(\overline{w}+b_1^B)}+e^{-r\overline{w}}+\beta V^B=0$$

I obtain:

$$e^{rb_2^*} = \frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\beta]e^{r\overline{w}}\overline{V}}$$

When $\overline{V}>V^C$, the optimal contract in this case is given by:

- The transfer at period t is b_2^C if the state of nature is 2.
- The transfer at period t is $b_1^*\geq b_1^C$ if the state of nature is 1 at period t and the state of nature has not been realized at any moment during the first $t-1$ periods.
- The transfer at period t is b_1^C if the state of nature is 1 at period t and the state of nature was realized at some point during the first $t-1$ periods.

To find b_1^*, I must isolate it in:

$$\overline{V} = \rho\left(-e^{-r(\overline{w}+b_1^*)}+e^{-r\overline{w}}+\beta\overline{V}\right)+(1-\rho)\left(-e^{-r(\overline{w}+b_2^C)}+e^{-r\overline{w}}+\beta V^C\right)$$

With some manipulations...

$$\begin{aligned}(1-\beta\rho)\overline{V} &= \rho\left(-e^{-r(\overline{w}+b_1^*)}+e^{-r\overline{w}}\right)+(1-\rho)\left(-e^{-r(\overline{w}+b_2^C)}+e^{-r\overline{w}}+\beta V^C\right)\\ (1-\beta\rho)e^{r\overline{w}}\overline{V} &= \rho\left(-e^{-rb_1^*}+1\right)+(1-\rho)\left(-e^{-rb_2^C}+1+\beta e^{r\overline{w}}V^C\right)\end{aligned}$$

$$\begin{aligned}(1-\beta\rho)e^{r\overline{w}}\overline{V}-(1-\rho)\left(-e^{-rb_2^A}+1+\beta e^{r\overline{w}}V^A\right) &= \rho\left(-e^{-rb_1^{sb}}+1\right)\\ 1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho}\left(-e^{-rb_2^A}+1+\beta e^{r\overline{w}}V^A\right) &= e^{-rb_1^{sb}}\end{aligned}$$

By the self-enforcing constraint of agent 2, I have $V^C=\frac{(1-\rho)\left(-e^{-r(\overline{w}+b_2^C)}+e^{-r\overline{w}}\right)}{1-\beta(1-\rho)}$.

$$\begin{aligned}e^{-rb_1^*} &= 1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho}\left(-e^{-rb_2^C}+1+\beta e^{r\overline{w}}\frac{(1-\rho)\left(-e^{-r(\overline{w}+b_2^A)}+e^{-r\overline{w}}\right)}{1-\beta(1-\rho)}\right)\\ e^{-rb_1^{sb}} &= 1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho}\left(-e^{-rb_2^C}+1+\beta\frac{(1-\rho)\left(-e^{-rb_2^C}+1\right)}{1-\beta(1-\rho)}\right)\\ e^{-rb_1^{sb}} &= 1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho(1-\beta+\beta\rho)}\left(1-e^{-rb_2^C}\right)\\ e^{rb_1^{sb}} &= \frac{\rho(1-\beta+\beta\rho)}{\rho(1-\beta+\beta\rho)+(1-\rho)\left(1-e^{-rb_2^C}\right)-(1-\rho\beta)(1-\beta+\beta\rho)*e^{rd}*\overline{V}}\end{aligned}$$

Let V^{MAX} be the maximal value for $\overline{V}$. Then, V^{MAX} is reached when $b_1^*=0$. To have $b_1^*=0$, I must have:

$$\begin{aligned}(1-\beta\rho)(1-\beta+\beta\rho)e^{r\overline{w}}V^{MAX} &= (1-\rho)\left(1-e^{-rb_2^C}\right)\\ V^{MAX} &= \frac{(1-\rho)e^{-r\overline{w}}}{(1-\beta\rho)(1-\beta+\beta\rho)}\left(1-e^{-rb_2^C}\right)\end{aligned}$$

I have already found that the Pareto frontier is composed of three parts. Let's begin with the second one, when the optimal contract is first-best. By Proposition 4, I know that the Pareto frontier is given by:

$$U^{fb} = \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right]-\frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)e^{r\overline{w}}\overline{V}}\right]$$

Then, when $\overline{V}\in\left[V^B,V^C\right]$, the Pareto frontier is given by this relation.

For the first case, i.e. when $\overline{V}\in\left[0,V^B\right]$, I can use the same approach from the preceding proof.

$$U^{OP}\left(\overline{V}\right)=\rho\left(-e^{-r(y_1-b_1^B)}+e^{-ry_1}+\beta U^B\right)+(1-\rho)\left(-e^{-ry_2}e^{rb_2^*}+e^{-ry_2}+\beta U^{OP}\left(\overline{V}\right)\right)$$

With

$$U^B=\rho\left(-e^{-r(y_1-b_1^B)}+e^{-ry_1}+\beta U^B\right)+(1-\rho)\left(-e^{-ry_2}e^{rb_2^B}+e^{-ry_2}+\beta U^B\right)$$

If I compute $U^{OP}-U^{B}$, I find that

$$\begin{aligned} U^{OP}\left(\overline{V}\right)-U^{B} &= (1-\rho)\left(-e^{-ry_2}\left(e^{rb_2^*}-e^{rb_2^B}\right)+\beta\left(U^{OP}-U^{B}\right)\left(\overline{V}\right)\right) \\ U^{OP}\left(\overline{V}\right)-U^{B} &= \frac{(1-\rho)\,e^{-ry_2}}{1-\beta+\rho}\left(e^{rb_2^B}-e^{rb_2^*}\right) \end{aligned}$$

Then,

$$\begin{aligned} U^{OP}\left(\overline{V}\right) &= \frac{(1-\rho)\,e^{-ry_2}}{1-\beta+\rho}\left(e^{rb_2^B}-e^{rb_2^*}\right) \\ &+\frac{\left(\rho\left(-e^{-r(y_1-b_1^B)}+e^{-ry_1}\right)+(1-\rho)\left(-e^{-ry_2}e^{rb_2^B}+e^{-ry_2}\right)\right)}{1-\beta} \end{aligned}$$

And if I replace $e^{rb_2^*}$ by $\frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\beta]e^{r\overline{w}}\overline{V}}$, I find that:

$$\begin{aligned} U^{OP}\left(\overline{V}\right) &= \frac{(1-\rho)\,e^{-ry_2}}{1-\beta+\rho}\left(e^{rb_2^B}-\frac{(1-\rho)}{(1-\rho)-\left[1-(1-\rho)\beta\right]e^{r\overline{w}}\overline{V}}\right) \\ &+\frac{\left(\rho\left(-e^{-r(y_1-b_1^B)}+e^{-ry_1}\right)+(1-\rho)\left(-e^{-ry_2}e^{rb_2^B}+e^{-ry_2}\right)\right)}{1-\beta} \end{aligned}$$

Now, for the case where $\overline{V}\in\left[V^{C},V^{MAX}\right]$. In this case, the utility function of agent 1 is given by:

$$U^{OP}\left(\overline{V}\right)=\rho\left(-e^{-r(y_1-b_1^*)}+e^{-ry_1}+\beta U^{OP}\left(\overline{V}\right)\right)+(1-\rho)\left(-e^{-r(y_2-b_2^C)}+e^{-ry_2}+\beta f\left(V^{C}\right)\right)$$

I already know that $-e^{-r(y_2-b_2^C)}+e^{-ry_2}+\beta U^{C}=0$. Then,

$$\begin{aligned} U^{OP}\left(\overline{V}\right) &= \frac{\rho}{1-\beta\rho}\left(-e^{-r(y_1-b_1^*)}+e^{-ry_1}\right) \\ U^{OP}\left(\overline{V}\right) &= \frac{\rho}{1-\beta\rho}e^{-ry_1} \\ &\left(1-\frac{\rho\,(1-\beta+\beta\rho)}{\rho\,(1-\beta+\beta\rho)+(1-\rho)\left(1-e^{-rb_2^C}\right)-(1-\rho\beta)\,(1-\beta+\beta\rho)\,e^{r\overline{w}}\overline{V}}\right) \end{aligned}$$

■

Of course, the Pareto frontier in each case is dominated by the Pareto frontier in the case without self-enforcing constraints.[12] Figures 7 and 8 illustrate this fact.

At the opposite of the case where no first-best contracts are self-enforcing, a part of the unconstrained Pareto frontier may be reached when some first-best contracts are self-enforcing. This comes from the fact that, if a first-best contract is self-enforcing, then self-enforcing constraints do not apply and the problem is similar to the one without self-enforcing constraints.

If I take a look at Figure 8, I see that the Pareto frontier reaches the unconstrained Pareto frontier at the middle. At the extremities, self-enforcing constraints apply and no first-best contracts are possible. The gain to respect the contract is not high enough to compensate agents to accept a net transfer to the other. In extremities, a NTSEC exists but it cannot be first-best.

[12] The Pareto frontier in case where some first-best contracts are self-enforcing is weakly dominated while the Pareto frontier in the other case is dominated everywhere.

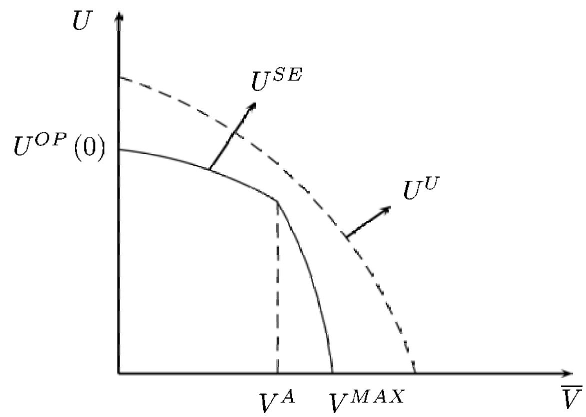

Figure 7: Pareto Frontier with no self-enforcing first-best contracts

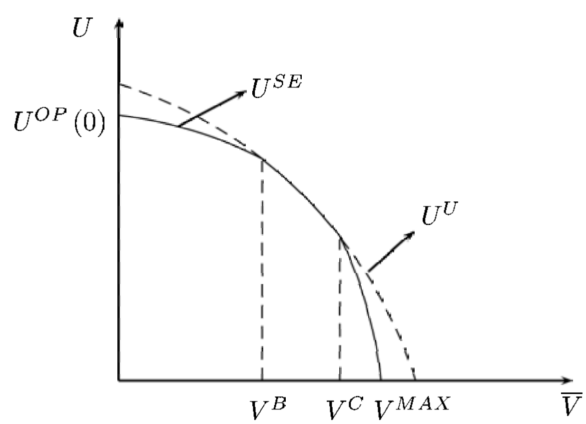

Figure 8: Pareto Frontier with self-enforcing first-best contracts

4 Variance

Thomas and Worrall [19] show that there exist 2 thresholds β_* and β^* with $0 < \beta_* < \beta^* < 1$ such that for any $\beta \in [0, \beta_*]$ the optimal contract is the TSEC; for any $\beta \in (\beta_*, \beta^*)$ the optimal contract is NTSEC but this contract is not first-best; and for $\beta \in [\beta^*, 1)$ some first-best contracts are self-enforcing. I now examine the effect of the variance on these thresholds.

To do so, I constrain our analysis to the case where agent 2 is risk-neutral. In this case, the problem can be written as

$$MAX \quad U(\delta, h_1) \tag{6}$$

subject to

$$\begin{aligned} V(\delta, h_1) &\geq \overline{V} \\ u(y_s - b_\tau(h_\tau, s)) - u(y_s) + \beta U(\delta, h_{\tau+1}) &\geq 0 \qquad \tau = 1, 2, ... \quad \forall s \in S, \quad \forall h_\tau \\ b_\tau(h_\tau, s) + \beta V(\delta, h_{\tau+1}) &\geq 0 \qquad \tau = 1, 2, ... \quad \forall s \in S, \quad \forall h_\tau \end{aligned}$$

Let $\mathbb{F}(F_1)$ be the set of all distribution functions for which the number of states of nature is equal to the number of states of nature of F_1 and the revenue in state s is given by $(y_s)_1 + \gamma((y_s)_1 - y)$ for $y = \mathrm{E}_s[(y_s)_1]$ and for all $\gamma > 0$. Note that for all distributions of revenue $F_2 \in \mathbb{F}(F_1)$, the expected revenue is equal to the expected revenue of F_1, in other words $\mathrm{E}_s[(y_s)_2] = \mathrm{E}_s[(y_s)_1]$. Distribution F_2 is a mean-preserving spread of distribution F_1.

Proposition 7 *Suppose I have two distributions of revenue, F_1 and $F_2 \in \mathbb{F}(F_1)$. Let $\overline{y}_1$ be the expected value of the revenue under F_1. Let $(\beta_*)_1$ and β_1^* be respectively the threshold to have a NTSEC and the threshold to have a first-best self-enforcing contract with the distribution of revenues F_1 and let $(\beta_*)_2$ and β_2^* be the thresholds with F_2. Then*

a) $(\beta_*)_1 > (\beta_*)_2$;

b) $\beta_1^* > \beta_2^*$.

Proof. a): Let $\beta > (\beta_*)_1$ and δ_1 be the optimal contract. Then I have for $t = 1, 2, ..., \forall s \in S$ and $\forall h_t$,

$$u\left((y_s)_1 - b_t^1(h_t, s)\right) - u((y_s)_1) + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[u\left(y_1^\tau - b_\tau(h_\tau, s)\right) - u(y_1^\tau)\right]\right] \quad \geq \quad 0$$

By strict concavity of u, then

$$u\left((y_s)_2 - b_t^1(h_t, s)\right) - u((y_s)_2) + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[u\left(y_2^\tau - b_\tau^1(h_\tau, s)\right) - u(y_2^\tau)\right]\right] > 0$$

Let $\widetilde{\delta}^2$ be the contract such that $\widetilde{b}_\tau^2(h_\tau, s) = b_\tau^1(h_\tau, s) + \epsilon$ with $\epsilon > 0$. By continuity, I know there exists an ϵ such that

$$\begin{aligned} u\left((y_s)_2 - b_t^2(h_t, s)\right) - u((y_s)_2) + \beta U\left(\delta^2, h_{t+1}\right) &> 0 \\ b_t^2(h_t, s) + \beta V\left(\delta^2, h_{t+1}\right) &> 0 \end{aligned}$$

Then, I can find a NTSEC for every $\beta > (\beta_*)_1$. Since u and v are strictly increasing, then $(\beta_*)_1 > (\beta_*)_2$.

b) Now, let δ_1 be the optimal first-best contract when the distribution of revenue is F_1 and $\beta = \beta_1^*$. Since δ_1 is first-best, then transfers are independent of the history. Let b_s^1 be the transfer in state s. By definition, if δ_1 is a first-best contract, the ratios of marginal utilities for agent 1 and agent 2 for each state must be equal.

$$\frac{u'\left((y_1)_2 - b_1^1\right)}{v'\left(\overline{w} + b_1^1\right)} = \frac{u'\left((y_2)_2 - b_2^1\right)}{v'\left(\overline{w} + b_2^1\right)} = \ldots = \frac{u'\left((y_S)_2 - b_S^1\right)}{v'\left(\overline{w} + b_S^1\right)}$$

If agent 2 is risk-neutral, then the first-best contract leaves agent 1 with a constant stream of net revenue,

$$(y_s)_1 - b_s^1 = (y_\sigma)_1 - b_\sigma^1 \quad \forall s, \sigma \in S \tag{7}$$

Let $\overline{b}_1$ be the expected value of the transfers under the distribution F_1. Consider the contract δ^2 where $b_s^2 = (1+\gamma)b_s^1$.

If I examine agent 2's self-enforcing constraints with the contract δ^2, I have that $\forall s \in S$,

$$b_s^2 + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} b_s^2\right]$$

If I replace b_s^2 with their values, I find

$$\begin{gathered}(1+\gamma)b_s^1 + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t}(1+\gamma)b_s^1\right] \\ (1+\gamma)\left(b_s^1 + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} b_s^1\right]\right)\end{gathered}$$

Since δ_1 is self-enforcing, then .

$$(1+\gamma)\left(b_s^1 + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} b_s^1\right]\right) \geq 0$$

If I examine agent 1's self-enforcing constraints under the distribution F_2, I have that $\forall s \in S$,

$$u\left((y_s)_2 - b_s^2\right) - u\left((y_s)_2\right) + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t}\left[u\left(y_2^\tau - b_s^2\right) - u\left(y_2^\tau\right)\right]\right]$$

If I replace $(y_s)_2$ and b_s^2 by their values, I find

$$\begin{gathered}u\left((y_s)_1 + \gamma\left((y_s)_1 - y\right) - (1+\gamma)b_s^1\right) - u\left((y_s)_1 + \gamma\left((y_s)_1 - y\right)\right) \\ +\beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t}\left[u\left(y_1^\tau + \gamma\left(y_1^\tau - y\right) - (1+\gamma)b_s^1\right) - u\left(y_1^\tau + \gamma\left(y_1^\tau - y\right)\right)\right]\right]\end{gathered}$$

$$u\left((1+\gamma)((y_s)_1 - b_s^1) - \gamma y\right) - u\left((1+\gamma)(y_s)_1 - \gamma y\right)$$
$$+\beta E_s^t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left[u\left((1+\gamma)(y_1^\tau - b_s^1) - \gamma y\right) - u\left((1+\gamma)y_1^\tau - \gamma y\right)\right]\right]$$

Because the self-enforcing constraints for agent 1 matter only when transfers are positive, which is the case when revenues are high, I concentrate my attention on those cases. Since $(1+\gamma)((y_s)_1 - b_s^1) - \gamma y < (y_s)_1 - b_s^1$ when $(y_s)_1 < y$ and $(1+\gamma)((y_s)_1 - b_s^1) - \gamma y > (y_s)_1 - b_s^1$ when $(y_s)_1 > y$, then, by the strictly concavity of u and since δ_1 is self-enforcing, I have that

$$u\left((1+\gamma)((y_s)_1 - b_s^1) - \gamma y\right) - u\left((1+\gamma)(y_s)_1 - \gamma y\right)$$
$$+\beta E_s^t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left[u\left((1+\gamma)(y_1^\tau - b_s^1) - \gamma y\right) - u\left((1+\gamma)y_1^\tau - \gamma y\right)\right]\right] > 0$$

By the same argument I use in a), there exists a $\epsilon > 0$ such that the contract δ_ϵ with $b_s^\epsilon = b_s^2 + \epsilon$, which is first-best, respects the self-enforcing constraint with strict inequality. ■

When the variance increases, the gain for agent 1 to sign a contract increases since agent 1 is risk-averse. Then, the incentive is bigger for agent 1 to sign a contract. Without the assumption about the type of change in agent 1's revenue, an increase in the variance does not necessarily result in a lower threshold.[13] It could be that the increase in the tails are so large that they cannot be compensated by other states of nature. Take the following example: Suppose that there are two revenue distributions F_1 and F_2. Let p_y^i be the probability to get y under the distribution function F_i. Suppose F_1 is characterized by $p_5^1 = p_{10}^1 = 0.5$. Suppose also that $p_5^2 = p_{10}^2 = 0.495$, $p_0^2 = 0.009925$ and $p_{1000}^2 = 0.000075$. It is easy show that the expected revenue is the same under F_1 and F_2 but the variance under F_2 is higher. The gain to break the contract when the revenue is 1000 could be positive for any possible contract and then, it is possible that, for a given discount factor β, there is a NTSEC for F_1 but not for F_2.

5 Merger

The question of mergers in the context of self-enforcing constraints is interesting. It has often been argued that conglomerates serve the purpose of providing insurance to shareholders. With the sophistication of financial markets, many have raised doubts about the ability of mergers for providing insurance beyond that which shareholders can get by themselves. This is certainly true in the presence of perfect financial markets. When these markets are imperfect, however, conglomerates may play a role. A merger could potentially provide better insurance than imperfect financial markets. I examine this logic when financial imperfections are caused by commitment problems, meaning that financial contracts must be self-enforcing.

In the previous section, I show that an increase in the variance decreases the threshold beyond which it is possible to sign a NTSEC. Proposition 7 gives the possibility to discuss mergers of firms with perfectly correlated revenues. If two firms have perfectly correlated revenues, then the

[13] It is possible to get this kind of result for the case where agent 2 has a random revenue but the condition over the increase in the variance does not stay the same. To obtain a result in the case of random revenue for both agents, I must define some conditions on revenues of both agents.

merged firm will have the same number of states of nature. By Proposition 7, if firm revenues are negatively correlated, then the merger decreases the variance and thresholds increase. But, since the merged firm has smoother post-merger revenue, the final effect is quite difficult to predict. In the case of perfect positive correlation, the merger increases the range of β's for which there exists a NTSEC. On the other hand, the variance of the revenues increases at the same time. Consequently, the ultimate impact of the merger on agent 1' utility is difficult to see. To get an idea about the possible outcomes, I use a numerical example.

I use a CARA function to model a risk-averse agent's utility and I suppose there are two symmetric risk-averse firms with random revenues. They have the possibility of signing a self-enforcing contract with a risk-neutral agent (the market). There are two states of nature with equal probability ($\frac{1}{2}$). In the bad state, firms get \$1 each and they get \$3 in the good state. Let the risk-aversion coefficient for both firms r equal to 1. Firm 1 has to choose between two possibilities: either stand alone to get financing, or to merge with another firm and then get financing.

5.1 Stand-alone case

Both firms are symmetric and thus I study the stand alone problem for one firm, say firm 1. Let x_1 and x_2 be firm 1's revenue in states 1 and 2 respectively and b_1 and b_2 the transfers. I assume that there are many risk-neutral agents. Consequently, the reservation value for them is 0 and I can write the stand-alone problem as follows:

$$MAX \quad E_s^1 \left[-e^{-r(x_s - b_1(h_1,s))} + e^{-rx_s} + \beta U(\delta, h_2) \right] \tag{8}$$

subject to

$$\begin{aligned} E_s^1 \left[b_1(h_1, s) + \beta V(\delta, h_2) \right] &\geq 0 \\ -e^{-r(x_s - b_\tau(h_\tau,s))} + e^{-rx_s} + \beta E_s^\tau [U(\delta, h_{\tau+1})] &\geq 0 \qquad \tau = 1, 2, ... \quad s = 1, 2 \quad \forall h_\tau \\ b_\tau(h_\tau, s) + \beta E_s^\tau [V(\delta, h_{\tau+1})] &\geq 0 \qquad \tau = 1, 2, ... \quad s = 1, 2 \quad \forall h_\tau \end{aligned}$$

Let U_{SA} be the expected utility for firm 1 in the stand-alone situation. I define the per period certainty equivalent (CE_{SA}) as the amount of money for which firm 1 is indifferent between this amount and its net revenue with the self-enforcing contract. In other words, the certainty equivalent in the stand-alone case is such that

$$-e^{-rCE_{SA}} = (1 - \beta) U_{SA}$$

Table 1 gives U_{SA} for different values of β. The thresholds to have a NTSEC and to have a self-enforcing first-best contract are approximately $\beta_* = 0.52$ and $\beta^* = 0.76$ respectively.

Table 1: Utility of firm 1 in the stand-alone case

β	U_{SA}	CE_{SA}	β	U_{SA}	CE_{SA}
0.20	-0.261042	1.5662	0.60	-0.460352	1.6921
0.22	-0.267735	1.5662	0.62	-0.466551	1.7300
0.24	-0.274781	1.5662	0.64	-0.475109	1.7659
0.26	-0.282207	1.5662	0.66	-0.486559	1.7992
0.28	-0.290046	1.5662	0.68	-0.501567	1.8295
0.30	-0.298333	1.5662	0.70	-0.520951	1.8561
0.32	-0.307108	1.5662	0.72	-0.545875	1.8783
0.34	-0.316414	1.5662	0.74	-0.577699	1.8958
0.36	-0.326302	1.5662	**0.76**	**-0.618338**	**1.9078**
0.38	-0.336828	1.5662	0.78	-0.669118	1.9159
0.40	-0.348055	1.5662	0.80	-0.730235	1.9238
0.42	-0.360057	1.5662	0.82	-0.805026	1.9317
0.44	-0.372917	1.5662	0.84	-0.898617	1.9395
0.46	-0.386728	1.5662	0.86	-1.019065	1.9472
0.48	-0.401602	1.5662	0.88	-1.179796	1.9549
0.50	-0.417667	1.5662	0.90	-1.404976	1.9626
0.52	**-0.435069**	**1.5662**	0.92	-1.742939	1.9702
0.54	-0.451988	1.5706	0.94	-2.306465	1.9777
0.56	-0.453389	1.6120	0.96	-3.433892	1.9852
0.58	-0.456081	1.6526	0.98	-6.816911	1.9926

Figure 9 graphs the certainty equivalent as a function of β. Note that there are two breakpoints. The first breakpoint is when β reaches 0.52. For all β lower than or equal to 0.52, there is no NTSEC. Agent 1 is unable to sign a contract which is non-trivial. Consequently, the per period utility remains unchanged while β increases but the certainty equivalent for the stand-alone case does not change with the value of β.[14] For greater values, some non-trivial contracts become self-enforcing, so the value for the certainty equivalent increases. The other breakpoint arrives at $\beta = 0.76$. At this point, the optimal self-enforcing contract converges to a first-best contract.

[14] U_{SA} changes since it's the weighted sum of present and future gains in utility.

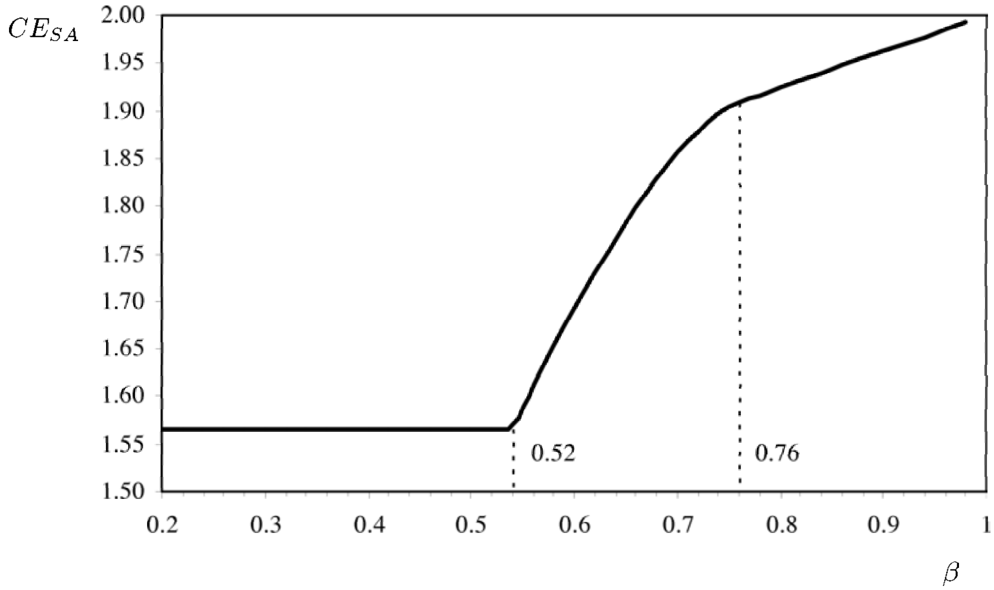

Figure 9: Certainty Equivalent

5.2 Merger case

The second possibility for firm 1 is to buy firm 2 by paying CE_{SA} in each period, and signing a self-enforcing contract considering that it gets the aggregate revenue. Since I have two states of nature for each firm, the merged firm will face four states of nature.

Table 2: States of nature

	state 1	state 2	state 3	state 4
firm 1 revenue	1	3	1	3
firm 2 revenue	1	1	3	3

To study the effect of correlation between firm revenues on the profitability of the merger, I need to define the coefficient of correlation ρ which is given by

$$\rho = \frac{COV(X,Y)}{\sigma_X \sigma_Y}$$

where σ_X and σ_Y are the standard error of revenues for firm 1 and firm 2 respectively.

Table 3 gives the probability of each state of nature for different coefficients of correlation.

Table 3: Coefficient of correlation and states of nature

ρ	state 1	state 2	state 3	state 4
-1	0	0.5	0.5	0
-0.8	0.05	0.45	0.45	0.05
-0.5	0.125	0.375	0.375	0.125
-0.2	0.2	0.3	0.3	0.2
0	0.25	0.25	0.25	0.25
0.2	0.3	0.2	0.2	0.3
0.5	0.375	0.125	0.125	0.375
0.8	0.45	0.05	0.05	0.45
1	0.5	0	0	0.5

Since there are two states of nature for each firm and they have symmetric payoffs, the merged firm faces three different states of nature. Let $z_1 = 2$, $z_2 = 4$ and $z_3 = 6$ be the revenues in each state. Using this approach allows for a simple model in which I can analyze the effect of correlation between firm revenues.

Let $b_t(h_t, s))$ be the transfer for period t in state s.[15] I suppose that the per period cost of acquiring firm 2 is its certainty equivalent (CE_{SA}). Then, the problem of the merged firm[16] is

$$MAX \quad E_s^1\left[-e^{-r(z_s - b_1(h_1,s)) - CE_{SA})} + e^{-r(z_s - CE_{SA})} + \beta U(\delta, h_2)\right] \tag{9}$$

subject to

$$E_s^1\left[b_1(h_1, s)) + \beta V(\delta, h_2)\right] \geq \overline{V}$$

and $\tau = 1, 2, ..., s = 1, 2, 3$ and $\forall h_\tau$,

$$\begin{aligned} -e^{-r(z_s - b_\tau(h_\tau,s)) - CE_{SA})} + e^{-r(z_s - CE_{SA})} + \beta E_s^\tau[U(\delta, h_{\tau+1})] &\geq 0 \\ b_\tau(h_\tau, s)) + \beta E_s^\tau[V(\delta, h_{\tau+1})] &\geq 0 \end{aligned}$$

The expected utility of the merged firm is given by U_M.

Table 4: Net gain of utility from the merger (positive value in bold)

β	$\rho = -1$	$\rho = -0.8$	$\rho = -0.5$	$\rho = -0.2$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.5$	$\rho = 0.8$	$\rho = 1$

Continued on next page

[15] Because there are more than 2 states of nature, the stationary contract is dependent on the history.

[16] With CARA utility functions, the payment of CE_{SA} does not affect the resolution of the problem. It is possible to isolate $e^{rCE_{SA}}$ in the objective function and in the firm self-enforcing constraints. Then, $e^{rCE_{SA}}$ affects only the utility but not the optimal contract itself.

Continued from previous page

β	$\rho=-1$	$\rho=-0.8$	$\rho=-0.5$	$\rho=-0.2$	$\rho=0$	$\rho=0.2$	$\rho=0.5$	$\rho=0.8$	$\rho=1$
0.20	**0.151**	**0.121**	**0.076**	**0.030**	**0**	-0.030	-0.076	-0.121	-0.151
0.22	**0.155**	**0.124**	**0.078**	**0.031**	**0**	-0.031	-0.078	-0.124	-0.155
0.24	**0.159**	**0.128**	**0.080**	**0.032**	**0**	-0.032	-0.080	-0.128	-0.158
0.26	**0.164**	**0.131**	**0.082**	**0.033**	**0**	-0.033	-0.082	-0.125	-0.143
0.28	**0.168**	**0.135**	**0.084**	**0.034**	**0**	-0.034	-0.084	-0.111	-0.127
0.30	**0.173**	**0.138**	**0.087**	**0.035**	**0**	-0.035	-0.073	-0.097	-0.110
0.32	**0.178**	**0.143**	**0.089**	**0.036**	**0**	-0.034	-0.060	-0.081	-0.093
0.34	**0.184**	**0.147**	**0.092**	**0.037**	**0**	-0.023	-0.046	-0.065	-0.076
0.36	**0.189**	**0.151**	**0.095**	**0.038**	**0.006**	-0.012	-0.032	-0.048	-0.058
0.38	**0.195**	**0.156**	**0.098**	**0.039**	**0.016**	**0.001**	-0.017	-0.031	-0.039
0.40	**0.202**	**0.162**	**0.101**	**0.045**	**0.027**	**0.014**	-0.001	-0.014	-0.021
0.42	**0.209**	**0.167**	**0.104**	**0.055**	**0.040**	**0.029**	**0.015**	**0.005**	-0.002
0.44	**0.216**	**0.173**	**0.108**	**0.066**	**0.053**	**0.044**	**0.032**	**0.023**	**0.018**
0.46	**0.224**	**0.180**	**0.112**	**0.079**	**0.067**	**0.059**	**0.049**	**0.038**	**0.037**
0.48	**0.233**	**0.186**	**0.117**	**0.092**	**0.083**	**0.076**	**0.068**	**0.055**	**0.056**
0.50	**0.242**	**0.194**	**0.127**	**0.107**	**0.099**	**0.093**	**0.078**	**0.072**	**0.075**
0.52	**0.252**	**0.202**	**0.139**	**0.122**	**0.116**	**0.111**	**0.092**	**0.088**	**0.094**
0.54	**0.261**	**0.208**	**0.149**	**0.136**	**0.131**	**0.116**	**0.102**	**0.101**	**0.109**
0.56	**0.245**	**0.187**	**0.132**	**0.121**	**0.117**	**0.094**	**0.080**	**0.082**	**0.093**
0.58	**0.228**	**0.166**	**0.115**	**0.106**	**0.087**	**0.070**	**0.057**	**0.061**	**0.075**
0.60	**0.212**	**0.143**	**0.098**	**0.092**	**0.062**	**0.044**	**0.029**	**0.037**	**0.055**
0.62	**0.195**	**0.120**	**0.082**	**0.061**	**0.034**	**0.015**	**0.001**	**0.045**	**0.035**
0.64	**0.178**	**0.096**	**0.067**	**0.034**	**0.004**	-0.019	-0.031	**0.024**	**0.015**
0.66	**0.161**	**0.071**	**0.053**	**0.004**	-0.029	-0.053	**0.012**	**0.005**	-0.004
0.68	**0.145**	**0.051**	**0.040**	-0.029	-0.066	-0.091	**0.001**	-0.014	-0.023
0.70	**0.130**	**0.034**	**0.031**	-0.063	-0.104	**0.001**	-0.016	-0.031	-0.039
0.72	**0.118**	**0.021**	-0.011	-0.100	**0**	-0.014	-0.031	-0.044	-0.052
0.74	**0.109**	**0.013**	-0.037	-0.138	-0.012	-0.025	-0.041	-0.054	-0.062
0.76	**0.104**	**0.012**	-0.063	-0.003	-0.018	-0.031	-0.046	-0.058	-0.065
0.78	**0.104**	**0.016**	-0.090	-0.006	-0.020	-0.032	-0.046	-0.058	-0.065
0.80	**0.103**	**0.021**	-0.123	-0.008	-0.022	-0.033	-0.047	-0.058	-0.063
0.82	**0.103**	**0.027**	-0.165	-0.011	-0.024	-0.035	-0.047	-0.057	-0.062
0.84	**0.102**	**0.034**	**0.013**	-0.014	-0.026	-0.036	-0.047	-0.057	-0.061
0.86	**0.102**	-0.010	**0.009**	-0.017	-0.028	-0.038	-0.048	-0.056	-0.061
0.88	**0.102**	-0.050	**0.004**	-0.020	-0.031	-0.039	-0.049	-0.056	-0.060
0.90	**0.101**	-0.107	-0.002	-0.024	-0.034	-0.041	-0.049	-0.055	-0.057
0.92	**0.101**	**0.033**	-0.009	-0.029	-0.037	-0.043	-0.050	-0.055	-0.058
0.94	**0.101**	**0.023**	-0.017	-0.034	-0.041	-0.046	-0.051	-0.055	-0.057
0.96	**0.100**	**0.008**	-0.027	-0.040	-0.045	-0.048	-0.052	-0.055	-0.056
0.98	**0.100**	-0.015	-0.039	-0.046	-0.049	-0.051	-0.053	-0.054	-0.055

Table 5 gives the value of the thresholds for each value of ρ. In the previous section, I find that the thresholds β_* and β^* must decrease (increase) while variance increases (decreases). Since the variance increases with the correlation coefficient, I have that thresholds decrease with ρ. These findings confirm the results of Proposition 7.

Table 5: Thresholds for NTSEC

β	$\rho = -0.8$	$\rho = -0.5$	$\rho = -0.2$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.5$	$\rho = 0.8$	$\rho = 1$	stand-alone
β_*	0.68	0.48	0.40	0.36	0.32	0.30	0.26	0.26	0.52
β^*	0.92	0.82	0.76	0.72	0.69	0.65	0.60	0.59	0.76

5.3 Results

Figures 10 and 11 show the differences in utility levels between the merger case with different correlations and the stand-alone case.

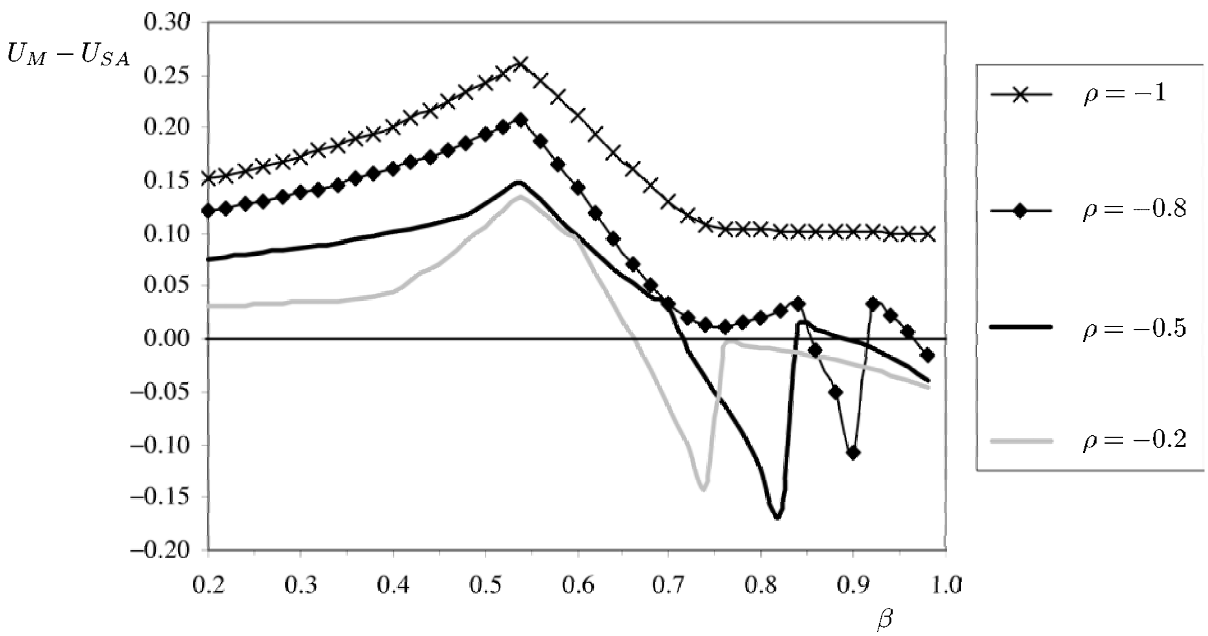

Figure 10: Negative Correlation and Merger

To analyze the effect of a merger, consider four cases: the perfect negative correlation case ($\rho = -1$), the negative (non perfect) correlation case ($\rho = -0.5$), the no correlation case ($\rho = 0$) and the positive (non perfect) correlation case ($\rho = 0.8$).

Case 1: The case of perfect negative correlation is represented by $\rho = -1$. This situation could arise when one firm has contracyclical revenues relative to the other one. Figure 12 shows

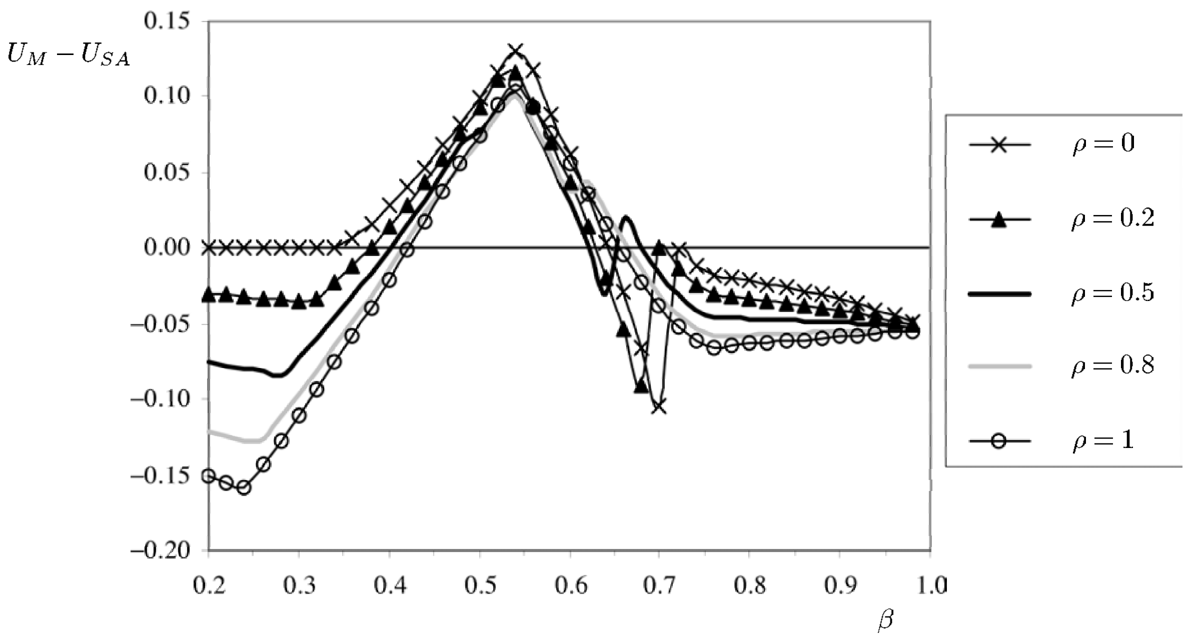

Figure 11: Positive Correlation and Merger

the certainty equivalent in the stand-alone case CE_{SA} and in the merger case CE_M with $\rho = -1$.

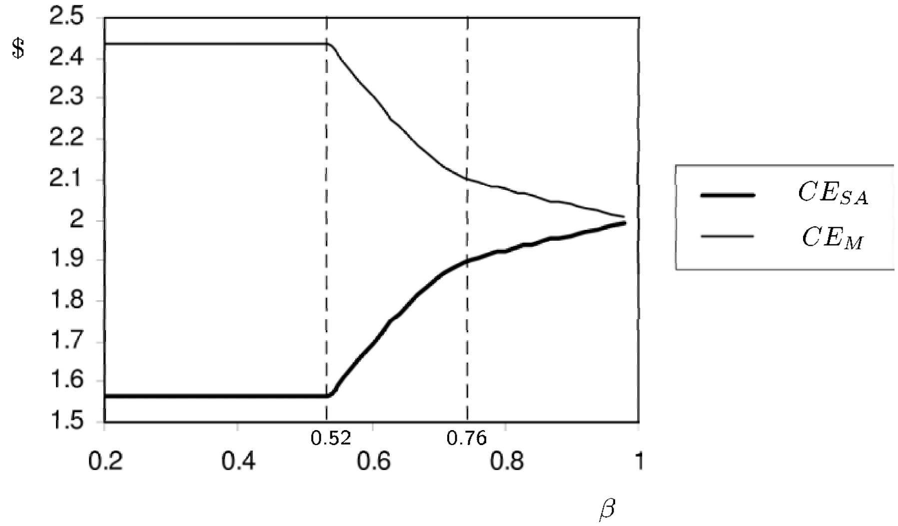

Figure 12: Certainty Equivalent for the stand-alone case and the merger case with $\rho = -1$

Note that the form of the certainty equivalent in both cases have the same form but inverse.

This particularity comes from the fact that, in the perfect negative correlation case, the firm revenue is constant for any given β. Consequently, there is no gain to sign a self-enforcing contract. However, CE_M is decreasing since firm 1 must pay CE_{SA} to firm 2. Since CE_{SA} depends on the value of β, the certainty equivalent for the merger case is decreasing with β but always greater than the certainty equivalent of the stand-alone case.

Case 2: When revenues are negatively, but not perfectly, correlated ($\rho = -0.5$), the benefit associated with a merger can be positive or negative depending on the value of β.

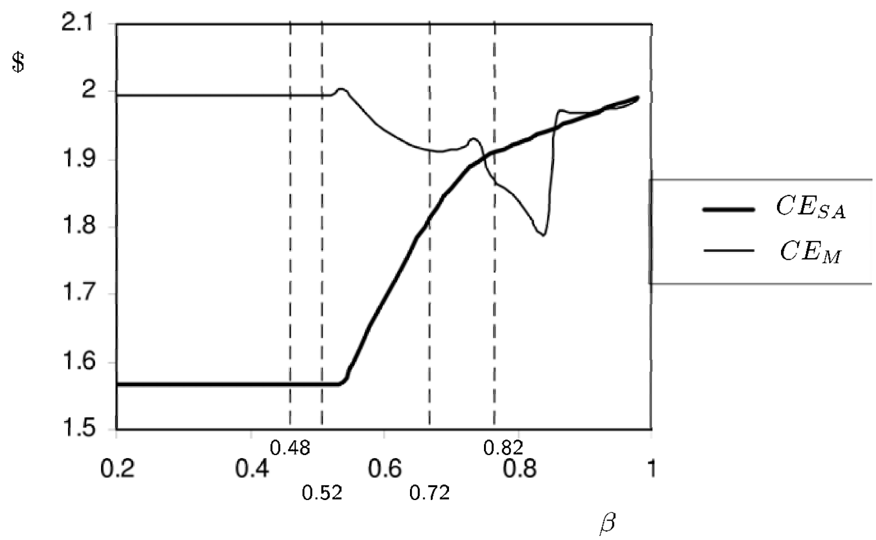

Figure 13: Certainty Equivalent for the stand-alone case and the merger case with $\rho = -0.5$

If β is lower than 0.48, there is no NTSEC for either the merged firm or the stand-alone firm, as there is for the stand-alone firm. But, the merged firm has a smoother revenue stream which leaves the firm with a gain by merging (see Figure 13). When β is between 0.48 and 0.52, it becomes possible for the merged firm to sign a NTSEC. The relative gain in utility becomes more important. At $\beta = 0.52$, it is possible for the stand-alone firm to sign a NTSEC. So the gain resulting from merging decreases and becomes negative at $\beta = 0.72$. For $\beta > 0.82$, it becomes possible for the merged firm to sign a first-best convergent contract. So the gain increases again with β but there is a threshold for which the gain cannot overcome the first-best convergent contract gain in the stand-alone case. After a small range of values for β (between 0.84 and 0.88) for which the merged firm gains, the net gain decreases and becomes negative.

What happens when β is close to 1 is another interesting case to study. When β is high enough, the merged firm and the stand-alone firm can sign a first-best convergent contract. Then, why does the merger appear non-profitable for β close to 1? First, by Proposition 3, if $\beta > \beta^*$, then the optimal contract converges monotonically to a first-best contract. Since I use the assumption that the reservation utility level for the market is equal to zero, the optimal contract, in both cases, converges to the first-best contract satisfying the self-enforcing constraints of the market. Let's suppose that δ^{SA} and δ^{M} are those first-best contracts. Then,

$$\begin{aligned} b_1^{SA} + \frac{\beta}{1-\beta} E_s[b_s^{SA}] &= 0 \\ b_1^{M} + \frac{\beta}{1-\beta} E_s[b_s^{M}] &= 0 \end{aligned}$$

where b_s^{SA} and b_s^{M} are respectively the transfer in state s for the stand-alone case and the merged case. I have already found (Equation (5)) that $b_z^i = b_s^i + y_z^\tau - y_s^t$ for $i = SA, M$. If I introduce these equations into the market self-enforcing constraints, I find:

$$\begin{aligned} b_1^{SA} + \frac{\beta}{1-\beta} E_s[b_1^{SA} + 10000 - x_s] &= 0 \\ b_1^{M} + \frac{\beta}{1-\beta} E_s[b_1^{SA} + 20000 - z_s] &= 0 \end{aligned}$$

and

$$\begin{aligned} b_1^{SA} + \frac{\beta}{1-\beta} (b_1^{SA} + 10000 - 20000]) &= 0 \\ b_1^{M} + \frac{\beta}{1-\beta} (b_1^{SA} + 20000 - 40000]) &= 0 \end{aligned}$$

I obtain that $b_1^M = 2b_1^{SA}$. This means that, once we subtract the CE_{SA}, the merged entity obtains the same level of utility than the stand-alone firm. Consequently, the optimal contracts of the merger case and the stand-alone case converge to first-best contracts that give the same level of utility.

Second, I know that optimal contracts are not first-best. They converge to some first-best contracts, but before state 1 is realized (see Section 3), transfers do not satisfy (5). Until then, the stand-alone firm gain more than the merged firm. Because of the concavity of CARA utility functions, the expected gain for being in the good state (state 2 for the stand-alone case and state 3 for the merger case) is higher in the stand-alone situation. It is therefore better for the firm to stand alone than to merge. This result applies to all cases where the correlation coefficient is not -1.[17]

Case 3: The independent case ($\rho = 0$) characterizes firms involved in different markets which are neither complements nor substitutes. In this case, there is no gain from merging when β is lower than 0.36. At this point, the merged firm can sign a NTSEC which leaves the firm better off. As for other cases, when β reaches 0.52, the gain from merging decreases. When β reaches 0.66, the net gain to merge becomes negative and remains negative while β increase. At $\beta = 0.72$, the merged firm can sign a first-best contract and the gain from merging increases but it is counterbalanced by the stand-alone contracting gain (see Figure

[17]When the correlation coefficient goes to -1, then the value of β such that to stand alone is better increases. For example, when $\rho = -0.9$, to stand alone is better when β is higher than 0.99.

14).

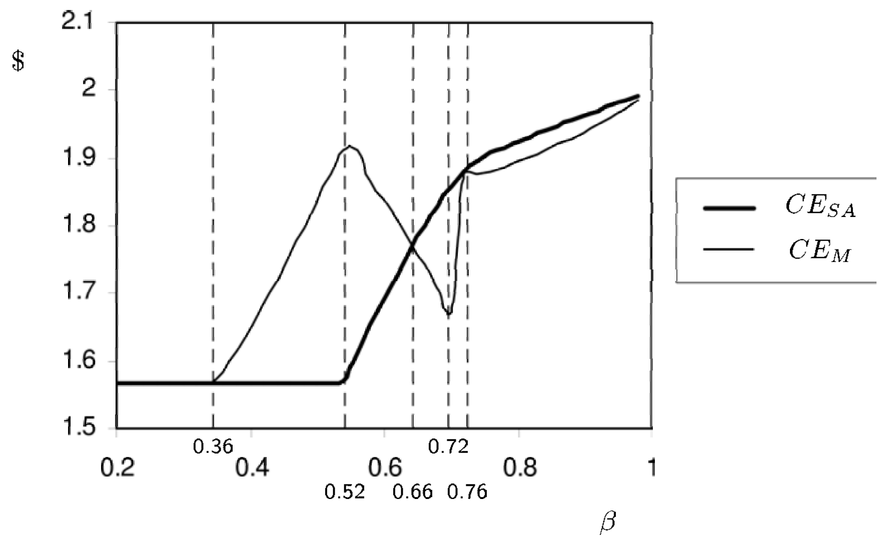

Figure 14: Certainty Equivalent for the stand-alone case and the merger case with $\rho = 0$

Case 4: The case where firms produce complements is represented by a positive correlation. With positive correlation ($\rho = 0.8$), the net gain from merging is negative for $\beta < 0.26$ (see Figure 15). At $\beta = 0.26$, the merged firm signs a NTSEC and the gain starts to increase. For β between 0.52 and 0.60, the gain diminishes as the stand-alone firm signs a NTSEC. For $\beta > 0.60$, the merged firm can sign a first-best contract. Consequently, the gain from the optimal first-best contract increases but the benefit to sign a contract for the stand-alone firm becomes more important, so the merger leaves more profits. Even with positively correlated revenues, there is an interval of β (in this case between 0.42 and 0.66) for which a merger could be profitable for the merged firm.

I can use the analysis I have from these different cases to draw general conclusions for the question of merger in a self-enforcing environment. If revenues are nearly perfectly negatively correlated, then the merger allows the new owner to smooth its revenues across time without any contract. This situation leads to the agent always being better off merging.

What is interesting is the influence of the correlation on the gain of a merger. When revenues are negatively correlated, the merger creates a kind of internal insurance market. The smoother revenue schedule leads to a gain in utility by decreasing the variance of revenues but decreases the possible gain from signing an insurance contract with the market. If *beta* is high but not too close of 1, then the merger could be beneficial. Take the case where $\rho = -0.8$. The merger option leaves the merged firm with gain when β is greater than 0.92 but smaller than 0.98. For all $\rho > -1$, then there exists a $\tilde{\beta} < 1$ such that for all $\beta \in [\tilde{\beta}, 1)$, then to stand alone is better for shareholders.

With no correlation, the new owner has the possibility of signing a contract in the case where β is small. Since the variance has increased, the possibility to sign a NTSEC has increased. But, the

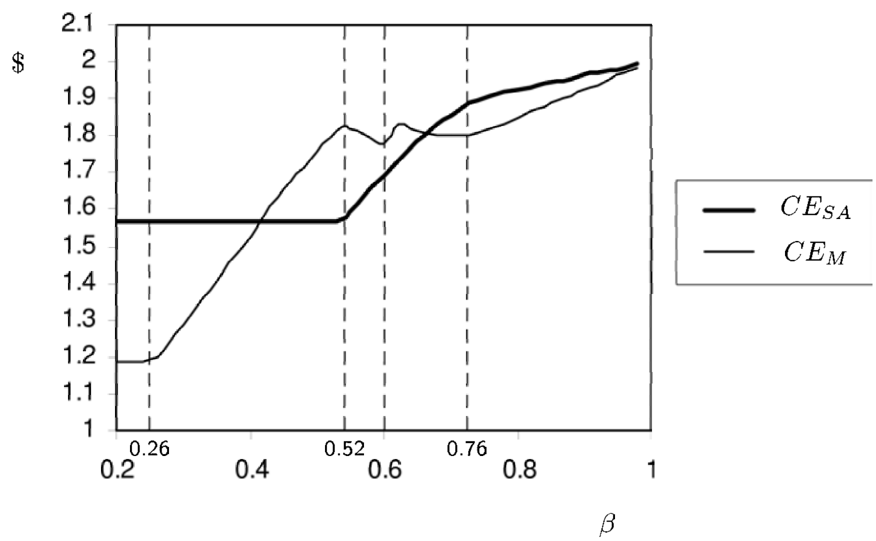

Figure 15: Certainty Equivalent for the stand-alone case and the merger case with $\rho = 0.8$

agent may do better in the stand-alone case depending of the value of β. As ρ goes to 1 (positive correlation), the threshold for having a self-enforcing contract decreases but it is possible that the gain from the contract cannot compensate the cost stemming from the increase of variance. So in the end, the agent is worse over for the majority of values of β.

6 Conclusion

In the first part of the paper, I explicitly solve the contract design problem with self-enforcing constraints. To obtain this solution, I must impose additional constraints on the model. The most important one is on the number of states of nature. The two states of nature problem is relatively easy to solve since there are only two transfers in the stationary contract. With three states, the number of transfers increases to four, and with four states, the number of transfers in the stationary contract is eight. The number of transfers in the stationary contract increases more quickly than the number of states of nature.

In the second part, I find that variance affects the nature of the contract. If the variance increases, then the potential benefits with respect to the contract increases and the threshold to have a NTSEC decreases.

The most interesting finding is the effect of self-enforcing constraints on the effects of a merger. I find that, even with a very high positive correlation between firms' revenues, there is some discount value for which firms could gain by a merger. The most important parameter in the merger decision seems to be the discount factor. If owners are not really patient, then a merger could lead to an increase in utility. This could explain in part why firms in the same market merge together while their revenues are highly positively correlated.

One of the possible avenues for future research would be to test the sensibility of these results to a change in the risk-aversion coefficient. My guess is that it will not change the scheme of the results but the level of thresholds.

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