Valuation of Credit Derivatives, and Credit Value-at-Risk, for the Energy Industry

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OVERVIEW

• Objective of Credit Value-at-Risk (CVAR): Quantifying and Optimizing Counterparty Credit Exposure, Accounting for both Market and Credit Risks

• CVAR I: Credit Exposure at Risk — Calculating the Magnitude of Counterparty Exposure

• CVAR II:
  1. Evaluating and Optimizing Counterparties Credit Line
  2. Quantifying Credit Risk Insurance
  3. Using CVAR to Evaluate a Differentially-Priced Trade with Two Counterparties of Different Credit Rating

• CVAR III: Credit Value-at-Risk

• CVAR IV: Combining Market and Credit Risks

• Summary
CVAR I: “Credit Exposure at Risk” — Evaluating Current and Prospective Counterparty Exposure

1. Model all relevant forward prices for given portfolio:

\[
\Delta F_i = F_i \sigma_i \epsilon_{it} \\
\text{Var}(\Delta F_i) = F_i^2 \sigma_i^2 t \\
\text{Cov}(\Delta F_i, \Delta F_j) = F_i F_j \rho_{ij} \sigma_i \sigma_j t
\]

where

\( F_i = \) price of \( i \)th futures contract \\
\( \sigma_{it} = \) volatility of \( i \)th futures contract for maturity \( t \) \\
\( \epsilon_{it} = \) is a Normally distributed random variable with zero mean and standard deviation equal to \( \sqrt{t} \) \\
\( \rho_{ij} = \text{Corr}(\ln F_i, \ln F_j) \)

2. Incorporate option contracts on forward/futures contract \( i \)

3. Compute portfolio \( \textit{basis} \) for the given counterparty:

\( n_i = \) number of futures contracts held long (and the counterparty is consequently short) \\
\( F_{i0} = \textit{basis} \) for \( i \)-th futures contract

\[
\text{Basis} = K = \sum_i n_i F_{i0} \\
\text{Current Portfolio Value} = P_0 = \sum_i n_i F_i
\]
Under full netting procedures, the current exposure $\Pi_0$ to the counterparty is

$$\Pi_0 = \sum_i n_i (F_i - F_{i0}) \equiv P_0 - K,$$  \hspace{1cm} (2)

of which the current credit risk exposure is given by

$$\text{max} \{P_0 - K, 0\}.$$ 

4. Model Forward Volatility and compute Portfolio Volatility to date $t$:

$$d\Pi = \sum_i N_i dF_i$$

$$\implies \text{Var} (\Delta \Pi) = \sum_i \sum_j N_i N_j F_i F_j \rho_{ij} \sigma_{it} \sigma_{jt} t \equiv \sigma_{Pt}^2 t$$

5. Compute Month $t$ Credit Exposure at Risk: Define for each counterparty the date $t$ Credit Exposure at Risk, CER$_t$:

$$\text{CER}_t = \begin{cases} 
\Pi_0 + \alpha \sigma_{Pt} \sqrt{t} & \text{if } \Pi_0 \geq 0 \\
\max \{\Pi_0 + \alpha \sigma_{Pt} \sqrt{t}, 0\} & \text{if } \Pi_0 < 0
\end{cases}$$ \hspace{1cm} (3)

6. Finally, define the counterparty’s exposure at risk, CER, as

$$\text{CER} \equiv \max_t \text{CER}_t,$$ \hspace{1cm} (4)

meaning that we would seek, for each counterparty, that date at which maximal 95th-percentile ($\alpha$) exposure currently occurs.

---

2When $\Pi_0 < 0$ — because the current counterparty exposure is in the counterpart's favor, i.e., $\Pi_0 \equiv \sum_i n_i (F_i - F_{i0}) < 0$ — then $\text{CER}_t = \max \{\Pi_0 + \alpha \sigma_{Pt} \sqrt{t}, 0\}$. 
CVAR I: Numerical Example

• Continuing with the previous numerical example, set
  \( N_1 = N_2 = 1; \sigma_1 = 30\%; \sigma_2 = 25\%; P_1 = $26; P_2 = $27 \)

• The dollar variance is
  \[
  \text{Var} (P_1 + P_2) = P_1^2 \sigma_1^2 + P_2^2 \sigma_2^2 + 2 \rho_{12} \sigma_1 \sigma_2 P_1 P_2 \\
  = 26^2 \times 0.3^2 + 27^2 \times 0.25^2 \\
  + 2 \times 0.6 \times 0.3 \times 0.25 \times 26 \times 27 = ($13.02)^2,
  \]
  with the dollar variance for the first month given by 13.02^2/12

• Consequently, at the 95-th percentile,
  \[
  \text{CER}_{1/12} = 1.645 \times 13.02 \times \sqrt{1/12} = $6.183
  \]

• Suppose now the term vols for the second month are
  \( \sigma_1 = 35\%; \sigma_2 = 30\% \),
respectively. Then the dollar variance for the first two-months is
  \[
  \text{Var} (P_1 + P_2) \Delta t = (P_1^2 \sigma_1^2 + P_2^2 \sigma_2^2 + 2 \rho_{12} \sigma_1 \sigma_2 P_1 P_2) \Delta t \\
  = (26^2 \times 0.35^2 + 27^2 \times 0.3^2 \\
  + 2 \times 0.6 \times 0.35 \times 0.3 \times 26 \times 27) (1/6) \\
  = ($15.39)^2 (1/6)
  \]
which in turns renders \( \text{CER}_{1/6} \) at the 95-th percentile equal to
  \[
  \text{CER}_{1/6} = 1.645 \times 15.39 \times \sqrt{1/6} = $10.34
  \]

• If, however, Contract 1 matures at month 1, then
  \[
  \text{Var} (P_2) \Delta t = (P_2^2 \sigma_2^2) \Delta t = (27^2 \times 0.3^2) (1/6) = ($8.16)^2 (1/6),
  \]
which in turn would render \( \text{CER}_{1/6} \) at the 95-th percentile equal to
  \[
  \text{CER}_{1/6} = 1.645 \times 8.16 \times \sqrt{1/6} = $5.48
  \]
CVAR I
Credit Exposure @ Risk – CP 1

Credit Exposure at Risk

Credit Value At Risk Calculations

Counterparty | Credit Rating | Expose Period | Recovery Rate | CE @ Risk
---|---|---|---|---
CP 1 | BBB | 10 | .5 | $459.09
CP 2 | BBB | 10 | .5 | $702.56

Commodity

Term Volatility

Credit Exposure

Process Date: 15-JAN-2001

Nucleus Software: A product of Caminus Corporation
CVAR I
CER Profile Graph – CP 1

Credit Exposure At Risk (CER)

Exposure Month

CER

$340.00

$360.00

$380.00

$400.00

$420.00

$440.00

$460.00

$480.00

2001 2

2001 3

2001 4

2001 5

2001 6

2001 7

2001 8

2001 9

2001 10

2001 11
### Credit Value At Risk Calculations

<table>
<thead>
<tr>
<th>Counterparty</th>
<th>Credit Rating</th>
<th>Expose Period</th>
<th>Recovery Rate</th>
<th>CE @ Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>C-P 3</td>
<td>A</td>
<td>11</td>
<td>.5</td>
<td>$2,124.58</td>
</tr>
<tr>
<td>C-P 1</td>
<td>BBB</td>
<td>10</td>
<td>.5</td>
<td>$459.09</td>
</tr>
<tr>
<td>C-P 2</td>
<td>BBB</td>
<td>10</td>
<td>.5</td>
<td>$702.56</td>
</tr>
</tbody>
</table>

### Probability

- Default Probability: .18
- Probability Volatility: .3

### Commodity

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Month</th>
<th>Pos</th>
<th>Price</th>
<th>Basis</th>
<th>Vol</th>
<th>Corr</th>
</tr>
</thead>
<tbody>
<tr>
<td>POWER</td>
<td>200106</td>
<td>40</td>
<td>6</td>
<td>1</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>POWER</td>
<td>200111</td>
<td>40</td>
<td>6</td>
<td>1</td>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>

### Term Volatility

<table>
<thead>
<tr>
<th>Month</th>
<th>Term Vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>200102</td>
<td>.587</td>
</tr>
<tr>
<td>200103</td>
<td>.629</td>
</tr>
<tr>
<td>200104</td>
<td>.683</td>
</tr>
<tr>
<td>200105</td>
<td>.741</td>
</tr>
<tr>
<td>200106</td>
<td>.8</td>
</tr>
</tbody>
</table>

### Credit Exposure

- 200102: $321.94
- 200103: $337.93
- 200104: $558.53
- 200105: $300.22
- 200106: $702.56
- 200107: $296.93
- 200108: $306.57
- 200109: $315.99

**Process Date:** 15-JAN-2001

Nucleus Software: A product of Caminus Corporation
Credit Exposure At Risk (CER)
CVAR I
Credit Exposure @ Risk – CP 3

Credit Value At Risk Calculations

<table>
<thead>
<tr>
<th>Counterparty</th>
<th>Credit Rating</th>
<th>Exposure Period</th>
<th>Recovery Rate</th>
<th>CE @ Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>C-P 3</td>
<td>A</td>
<td>11</td>
<td>.5</td>
<td>$2,124.68</td>
</tr>
<tr>
<td>C-P 1</td>
<td>BBB</td>
<td>10</td>
<td>.5</td>
<td>$459.09</td>
</tr>
<tr>
<td>C-P 2</td>
<td>BBB</td>
<td>10</td>
<td>.5</td>
<td>$702.56</td>
</tr>
</tbody>
</table>

Probability
- Default Probability: 0.05
- Probability Volatility: 2

Commodity

<table>
<thead>
<tr>
<th>Com</th>
<th>Month</th>
<th>Pos</th>
<th>Price</th>
<th>Basis</th>
<th>Vol</th>
<th>Corr</th>
</tr>
</thead>
<tbody>
<tr>
<td>GAS</td>
<td>200102</td>
<td>10</td>
<td>4.5</td>
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<td>6</td>
<td>.5</td>
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<tr>
<td>GAS</td>
<td>200103</td>
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<td>4.5</td>
<td>1</td>
<td>6</td>
<td>.5</td>
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<tr>
<td>GAS</td>
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<td>25</td>
<td>5.5</td>
<td>1</td>
<td>6</td>
<td>.5</td>
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<tr>
<td>GAS</td>
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<td>25</td>
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<td>1</td>
<td>6</td>
<td>.5</td>
</tr>
<tr>
<td>GAS</td>
<td>200106</td>
<td>30</td>
<td>7.5</td>
<td>1</td>
<td>6</td>
<td>.5</td>
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<tr>
<td>GAS</td>
<td>200107</td>
<td>40</td>
<td>8.5</td>
<td>1</td>
<td>8</td>
<td>.5</td>
</tr>
<tr>
<td>GAS</td>
<td>200108</td>
<td>40</td>
<td>8.5</td>
<td>1</td>
<td>8</td>
<td>.5</td>
</tr>
<tr>
<td>GAS</td>
<td>200109</td>
<td>30</td>
<td>8.5</td>
<td>1</td>
<td>6</td>
<td>.5</td>
</tr>
</tbody>
</table>

Term Volatility

<table>
<thead>
<tr>
<th>Month</th>
<th>Term Vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>200102</td>
<td>.545</td>
</tr>
<tr>
<td>200103</td>
<td>.686</td>
</tr>
<tr>
<td>200104</td>
<td>.836</td>
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<tr>
<td>200105</td>
<td>.689</td>
</tr>
<tr>
<td>200106</td>
<td>.744</td>
</tr>
<tr>
<td>200107</td>
<td>.8</td>
</tr>
<tr>
<td>200108</td>
<td>.8</td>
</tr>
<tr>
<td>200109</td>
<td>.8</td>
</tr>
</tbody>
</table>

Credit Exposure

<table>
<thead>
<tr>
<th>Month</th>
<th>Exposure</th>
</tr>
</thead>
<tbody>
<tr>
<td>200102</td>
<td>$2,124.68</td>
</tr>
<tr>
<td>200103</td>
<td>$2,102.42</td>
</tr>
<tr>
<td>200104</td>
<td>$2,025.84</td>
</tr>
<tr>
<td>200105</td>
<td>$1,880.88</td>
</tr>
<tr>
<td>200106</td>
<td>$1,697.04</td>
</tr>
<tr>
<td>200107</td>
<td>$1,413.60</td>
</tr>
<tr>
<td>200108</td>
<td>$897.13</td>
</tr>
<tr>
<td>200109</td>
<td>$475.76</td>
</tr>
</tbody>
</table>

Process Date: 15-JAN-2001
Compute
Display Chart

Nucleus Software: A product of Caminus Corporation
Intuitive Appeal of “Credit Exposure at Risk”

- The advantages of CER thus defined is that it captures three important features:
  1. The current “moneyness” of the exposure, $\Pi$
  2. The riskiness of that exposure, $\sigma_{Pt}$
  3. The effect of time $t$

thus justifying the definition of the term as “Credit Exposure at Risk.”

- For counterparties with identical credit risk, eq. (4)’s CER constitutes the mechanism for credit allocation across counterparties: Specifically, for counterparties assessed to have the same level of credit quality, the CER should be equalized across these clients.

- In sum, the inputs required for the Credit Exposure at Risk model include:
  1. **Financial Environment:** Prices of futures contracts; prices of option contracts; times to maturity of futures and options; volatilities of futures contracts; correlations across futures contracts
  2. **Counterparty Data:** Number of futures contracts; number and exercise prices of option contracts; initial values of futures and option contracts
An (Important and Relevant) Interpretation of the Black-Scholes Formula

• As is well-known, the payoff on a call option is
  \[ \max\{S - K, 0\} . \]

• The value of an option today, \( T \) years prior to expiration, is given by the discounted present value of the expectation \( E(\cdot) \):

\[
e^{-rT} E(\max\{S - K, 0\}) ,
\]

where the result of that expectation is the Black-Scholes formula:

\[
e^{-rT} E(\max\{S - K, 0\}) = SN(d) - Ke^{-rT} N(d - \sigma \sqrt{T})
= \text{Black-Scholes Formula}
\]

This result is valuable, in that it permits an evaluation of the value of credit insurance, and its dependence on the

• Moneyness,
• Volatility, and
• Default sensitivity

of the counterparty portfolio

\[ ^{3}\text{Technically, this expectation obtains under the so-called “risk-neutral” distribution.} \]
CVAR II: 1. Evaluating and Optimizing Counterparties Credit Line
2. Quantifying Credit Risk Insurance

1. As before, assume we have calculated

\[
\text{Var} (\Delta P) = \sum_i \sum_j N_i N_j F_i F_j \rho_{ij} \sigma_{it} \sigma_{jt} t.
\]

Make the transition to a LogNormal distribution solving for \( \sigma_{Pt} \) from

\[
\text{Var} (\Delta P) = \sigma_{Pt}^2 P^2 t
\]

2. Let \( p \) be defined as the marginal probability to default at a given point in time in the future, conditional on not having previously defaulted. Suppose that probability to default follows its own diffusion process,

\[
\frac{dp}{p} = \sigma_p dz_p,
\]

that is, \( p \) is Log Normally distributed \( (5) \)

where we allow for a correlation between \( \Pi_0 \) and \( p \):

\[
\rho \equiv \text{Corr} (dz_p, dz_P)
\]

3. Conditional on the event of default at date \( t \), let the recovery rate per promised date \( t \) dollar be given by the known parameter value of \( \alpha \) (which parameter can be made random, or dependent on the ratings category, as in \( \alpha_{\text{ratings category}} \))
The Normal and LogNormal Distributions: An Interpretation using the Numerical Example (Cont’d)

- Consider the previous analysis that pertained to a portfolio composed of Contract 1 and Contract 2
- Although the sum (a portfolio) of LogNormal random variables is not LogNormal, we approximate that portfolio as LogNormal
  - What is the dollar variance of a portfolio of $P_1$ and $P_2$?

$$\text{Var}(P_1 + P_2) = P_1^2 \sigma_1^2 + P_2^2 \sigma_2^2 + 2 \rho_{12} \sigma_1 \sigma_2 P_1 P_2$$

$$= 26^2 \times 0.3^2 + 27^2 \times 0.25^2$$

$$+ 2 \times 0.6 \times 0.3 \times 0.25 \times 26 \times 27 = (\$13.02)^2$$

- What is the value of the portfolio?

$$\Pi = P_1 + P_2 = 26 + 27 = \$53$$

- Consequently, what LogNormal variance is implied?

$$(P_1 + P_2)^2 \sigma^2_{\Pi} = P_1^2 \sigma_1^2 + P_2^2 \sigma_2^2 + 2 \rho_{12} \sigma_1 \sigma_2 P_1 P_2$$

$$\implies \sigma_{\Pi} = \sqrt{\frac{P_1^2 \sigma_1^2 + P_2^2 \sigma_2^2 + 2 \rho_{12} \sigma_1 \sigma_2 P_1 P_2}{(P_1 + P_2)^2}}$$

$$= 24.57\%$$
CVAR II: Counterparty Credit Line and Credit Risk Insurance (Cont’d)

4. In CVAR II, we extend the analysis of CVAR I by considering, for each of the firm’s counterparties, the (random) exposure given by

\[
\begin{cases} 
\text{Default, Partial Recovery at rate } \alpha & \text{with probability } \tilde{p}_t \\
0 & \text{with probability } 1 - \tilde{p}_t 
\end{cases}
\]

5. Let \( V_t \) be the current value of that risk exposure. Then,

\[
V_t = (1 - \alpha) e^{-rt} E \left[ \max \{ p_t (P_t - K), 0 \} \right] \\
= (1 - \alpha) e^{-rt} p_t \left[ P_0 e^{\rho \sigma_p t} \sigma_{P_t} N(d) - K N \left( d - \sigma_{P_t} \sqrt{t} \right) \right] \\
\]

where

\[
E(pP) = E(p) E(P) + \text{Cov}(p, P) = p_t P_0 e^{\rho \sigma_p t} \sigma_{P_t} \\
\]

\[
d \equiv \frac{\log(P_0/K) + \rho \sigma_p t \sigma_{P_t}}{\sigma_{P_t} \sqrt{t}} + \frac{1}{2} \sigma_{P_t} \sqrt{t}
\]

Scaled up by \( 1 - \alpha \), \( V_t \) is simply the current value of an option with a future random payoff \( \tilde{p}_t \tilde{P}_t \) and a random exercise price \( \tilde{p}_t K \).

6. With the above, we compute

\[
\sum_t V_t, \\
\]

where the \( \sum_t \) operator captures the probability and cost of default across all maturities.

This has the dual interpretation of:

(a) The number, across all ratings category, to be set as the appropriate credit line for the firm’s counterparties

(b) The no-arbitrage value of default-risk insurance for the specific counterparty

\[\text{With no default, the cost of the exposure is zero.}\]
Numerical Example of CVAR II

Assume the data of the previous numerical example. In addition, assume:

\[ T = 1 \]: Maturity of futures contract with Counterparty 1  
\[ r = 5\% \]: One-year riskfree rate of interest  

Now consider the following four possible states of nature at the end of the year:

<table>
<thead>
<tr>
<th>Case</th>
<th>Futures Contract Value</th>
<th>Default</th>
<th>Recovery Rate</th>
<th>Probability</th>
<th>Payoff on Default Insurance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$33</td>
<td>No</td>
<td>NA</td>
<td>45%</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$33</td>
<td>Yes</td>
<td>45%</td>
<td>5%</td>
<td>0.55 \cdot (33 - 26)</td>
</tr>
<tr>
<td>3</td>
<td>$20</td>
<td>No</td>
<td>NA</td>
<td>47%</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$20</td>
<td>Yes</td>
<td>NA</td>
<td>3%</td>
<td>0</td>
</tr>
</tbody>
</table>

Notes:
1. This is the futures contract value at its maturity date.
2. NA – Not Applicable, either because of No Default, or because Default occurred when Counterparty 1 had a positive, rather than negative, mark-to-market on the contract.
3. Note that default probability is positively correlated with Price: Default is more likely to occur when the Counterparty’s mark-to-market is negative (5\%), than when it is positive (3\%).
4. Payoff is positive only if default occurs when Counterparty 1 has a negative MTM on the contract.

Based on Table 1 and the one-year 5\% rate of interest, the value of CVAR II is:

\[
\text{CVAR II} = \text{Prob. of Default} \times \left( \frac{\text{Payoff in the Event of Default}}{1 + \text{Discount Rate}} \right)
\]

\[
= 0.05 \times \frac{0.55 \cdot (33 - 26)}{1.05} = \$0.1833. \tag{8}
\]
CVAR II: Estimating the Default Probablility Process
Direct Estimates of Default-Rate Volatilities

• Recall that the fundamental Default Prob. Equation is:

\[ \ln p_t \sim N (\ln \bar{p}_t, \sigma_{pt}), \]  

(9)

where the potential time-dependence enters through \( \ln \bar{p}_t \) and \( \sigma_{pt} \), and

where the volatility \( \sigma_{pt} \) should be evaluated as of time 0.

• In CreditRisk+, Credit Suisse First Boston provides a table reporting one-

year default rate and volatilities (Section 2.6.4):

<table>
<thead>
<tr>
<th>Credit Rating</th>
<th>One-Year Default Rate (%)</th>
<th>Average (( \mu ))</th>
<th>Standard Deviation (( \sigma ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.00</td>
<td></td>
<td>0.0</td>
</tr>
<tr>
<td>Aa</td>
<td>0.03</td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>A</td>
<td>0.01</td>
<td></td>
<td>0.0</td>
</tr>
<tr>
<td>Baa</td>
<td>0.12</td>
<td></td>
<td>0.3</td>
</tr>
<tr>
<td>Ba</td>
<td>1.36</td>
<td></td>
<td>1.3</td>
</tr>
<tr>
<td>B</td>
<td>7.27</td>
<td></td>
<td>5.1</td>
</tr>
</tbody>
</table>

1. Interpret the above statistics as evolving from a Normal, rather than LogNormal, distribution: \( p \sim N (\mu, \sigma) \)

2. To convert these to the LogNormal (9) distribution we are using, note that the Normal distribution is equivalent to \( \text{Var}(dp) = \sigma^2 \), and the LogNormal to \( \text{Var}(dp/p) = \sigma^2_{pt} \). Thus, the approximation we can use is:

\[ \sigma^2_{pt} \equiv \text{Var} \left( \frac{dp}{p} \right) = \left( \frac{\sigma}{p} \right)^2 \cong \left( \frac{\sigma}{\mu} \right)^2. \]

3. Thus, the \( \sigma_{pt} \) is given by the ratio of the third column in the table above divided by the second column.
CreditRisk$^+$ provides a table reporting recovery rates by seniority and security (Section 2.6.5):

<table>
<thead>
<tr>
<th>Seniority and Security</th>
<th>Average (%)</th>
<th>Deviation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior Secured Bank Loan</td>
<td>71.18</td>
<td>21.09</td>
</tr>
<tr>
<td>Senior Secured Public</td>
<td>63.45</td>
<td>26.21</td>
</tr>
<tr>
<td>Senior Unsecured Public</td>
<td>47.54</td>
<td>26.29</td>
</tr>
<tr>
<td>Senior Subordinated Public Debt</td>
<td>38.28</td>
<td>24.74</td>
</tr>
<tr>
<td>Subordinated Public Debt</td>
<td>28.29</td>
<td>20.09</td>
</tr>
<tr>
<td>Junior Subordinated Public Debt</td>
<td>14.66</td>
<td>8.67</td>
</tr>
</tbody>
</table>

Source: Historical Default Rates of Corporate Bond Issuers, 1920 - 1996 (January 1997), Moody’s Investors Service Global Credit Risk
Computing CVAR II: Continuing Numerical Example

- **Input Data:**

<table>
<thead>
<tr>
<th>Definition</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time to Expiration</td>
<td>$T$</td>
<td>1</td>
</tr>
<tr>
<td>Exercise Price</td>
<td>$K$</td>
<td>$53$</td>
</tr>
<tr>
<td>Adjusted Forward Price</td>
<td>$F e^{\rho \sigma_p \sigma P t}$</td>
<td>$55.40$</td>
</tr>
<tr>
<td>Forward Volatility</td>
<td>$\sigma_P$</td>
<td>24.57%</td>
</tr>
<tr>
<td>Expected Default Rate</td>
<td>$p$</td>
<td>7.5%</td>
</tr>
<tr>
<td>Default Volatility Rate</td>
<td>$\sigma_p$</td>
<td>30%</td>
</tr>
<tr>
<td>Corr(Default, Prices)</td>
<td>$\rho$</td>
<td>0.6</td>
</tr>
<tr>
<td>Recovery Rate</td>
<td>$\alpha$</td>
<td>0.45</td>
</tr>
<tr>
<td>Riskfree Rate</td>
<td>$r$</td>
<td>5%</td>
</tr>
</tbody>
</table>

- **Option Value:**

$$C \left( F = 55.40, \ K = 53, \ \sigma = 24.57\%, \ r = 5\%, \ T = 1 \right) = 6.26$$

- **CVAR II Value is**

$$\left( 1 - \alpha \right) pC = 0.55 \times 0.075 \times 6.26 = 0.2584$$
## Additional Application of CVAR II

Using CVAR II to Evaluate a Differentially-Priced Trade with Two Counterparties

<table>
<thead>
<tr>
<th>Variable</th>
<th>Counterparty A</th>
<th>Counterparty B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time to maturity</td>
<td>$t = 1$</td>
<td></td>
</tr>
<tr>
<td>Counterparty offer price</td>
<td>$F_{A0} = 95$</td>
<td>$F_{B0} = 96$</td>
</tr>
<tr>
<td>Adjusted futures price $P_0 e^{\rho \sigma Pt}$</td>
<td>98.19</td>
<td>96.11</td>
</tr>
<tr>
<td>Riskfree rate</td>
<td>$r = 5%$</td>
<td></td>
</tr>
<tr>
<td>Mid-point bid/ask</td>
<td>$F = 90$</td>
<td></td>
</tr>
<tr>
<td>Futures vol</td>
<td>$\sigma_F = 55%$</td>
<td></td>
</tr>
<tr>
<td>$1 - $ Recovery rate</td>
<td>$\alpha = 0.45$</td>
<td></td>
</tr>
<tr>
<td>Default prob</td>
<td>$p_A = 10%$</td>
<td>$p_B = 1%$</td>
</tr>
<tr>
<td>Default vol</td>
<td>$\sigma_{pA} = 20%$</td>
<td>$\sigma_{pB} = 2%$</td>
</tr>
<tr>
<td>Default corr</td>
<td>$\rho_A = 0.3$</td>
<td>$\rho_A = 0.1$</td>
</tr>
<tr>
<td>Total vol</td>
<td>$\sigma_{tA} = 63.91%$</td>
<td>$\sigma_{tB} = 57.0%$</td>
</tr>
<tr>
<td>Insurance cost, $V_t$</td>
<td>1.459</td>
<td>0.126</td>
</tr>
<tr>
<td>Market + Insurance Costs</td>
<td>6.459</td>
<td>6.126</td>
</tr>
</tbody>
</table>
CVAR III: Credit Value-at-Risk

• With $\sum_t V_t$ from CVAR II, we have

$$
\sum_t dV_t = P_0 \left( \sum_t \frac{\partial V_t}{\partial P_0} \right) \frac{dP_0}{P_0} + \sum_t \frac{\partial V_t}{\partial p_t} p_t \frac{dp_t}{p_t}
$$

(10)

where

$$
\frac{\partial V_t}{\partial P_0} = (1 - \alpha) p_t e^{\rho_{Pp}\sigma_{pt}\sigma_{pt}} N(d)
$$

$$
\frac{\partial V_t}{\partial p_t} = \frac{V_t}{p_t}
$$

• For confidence level $\beta$ (e.g., $\beta = 1.65$), the Value-at-Risk we seek will be given by $\beta \sqrt{\text{Var} (dC) \Delta t}$. In turn,

$$
\text{Var} \left( \sum_t dV_t \right) = P_0^2 \left( \sum_t \frac{\partial V_t}{\partial P_0} \right)^2 \sigma_{P0}^2 dt + \sum_t \left( \frac{\partial V_t}{\partial p_t} p_t \right)^2 \sigma_{pt}^2 dt
$$

$$
+ 2 \sum_t \sum_{\tau > t} \frac{\partial V_t}{\partial p_t} \frac{\partial V_{\tau}}{\partial p_{\tau}} p_{t} p_{\tau} \text{Cov} (\ln p_t, \ln p_{\tau}) dt
$$

$$
+ 2 \sum_t \sum_{\tau > t} \frac{\partial V_t}{\partial P_0} \frac{\partial V_{\tau}}{\partial p_{\tau}} P_0 p_{\tau} \text{Cov} (\ln P_0, \ln p_{\tau}) dt
$$

(11)

where

$$
\sigma_{P0} = \text{is the instantaneous, not term, vol of } P_0
$$

$$
\text{Cov} (\ln p_t, \ln p_{\tau}) = \rho_{pt, pt} \sigma_{pt} \sigma_{pt}
$$

$$
\text{Cov} (\ln P_0, \ln p_{\tau}) = \rho_{Pp} \sigma_{P0} \sigma_{pt}
$$

• Interpretation of CVAR III: CVAR III quantifies — at, say, the 95-th percentile — the potential deterioration of the value of CVAR II’s $C$ over the (exogenously specified) holding-period $\Delta t$
CVAR III: Numerical Example

• Implementing the $\text{Var}(\Delta V_t)$ equation, we have:

\[
\frac{\partial V_t}{\partial P_0} = (1 - \alpha) \, p \, \Delta C = 0.55 \times 0.075 \times 0.5889 = 0.0243
\]

\[
\frac{\partial V_t}{\partial p_t} = (1 - \alpha) \, C = 0.55 \times 11.13 = 6.12
\]

\[
\text{Var} (\Delta V_t) = P_0^2 \left( \frac{\partial V_t}{\partial P_0} \right)^2 \sigma_{P_0}^2 + \left( \frac{\partial V_t}{\partial p_t} \, p_t \right)^2 \sigma_{p_t}^2
\]

\[
+ 2 \frac{\partial V_t}{\partial P_0} \frac{\partial V_t}{\partial p_t} \, P_0 p_t \, \text{Cov} (\ln P_0, \ln p_t)
\]

\[
= 53^2 \times 0.0243^2 \times 0.2457^2 + (3.445 \times .075)^2 \times 0.3^2
\]

\[
+ 2 \times 0.6 \times 53 \times 0.0243 \times 0.2457 \times 3.445 \times .075 \times 0.3
\]

\[
= (0.368)^2
\]

• For a one-week holding period $\Delta t = 1/52$, CVAR III Value is

\[
\beta \sqrt{\text{Var} (\Delta V_t)} \, \Delta t = 1.645 \times 0.368 / \sqrt{52} = $0.084
\]
CVAR IV: VAR Combining Market and Credit Risks

- Consider the case of a portfolio containing a single forward contract:

<table>
<thead>
<tr>
<th>Type of VAR</th>
<th>Analytical Expression</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market (Traditional VAR)</td>
<td>$1.645 \times \sqrt{\text{Var}(\Delta F_1)/52}$</td>
<td></td>
</tr>
<tr>
<td>Credit VAR (CVAR III)</td>
<td>$1.645 \times \sqrt{\text{Var}(\Delta V)/52}$  $V$ is the value of CVAR II</td>
<td></td>
</tr>
<tr>
<td>CVAR IV</td>
<td>$1.645 \times \sqrt{\text{Var}(\Delta V - \Delta F_1)/52}$</td>
<td></td>
</tr>
</tbody>
</table>

- Why Var $(\Delta V - \Delta F_1)$?
  Consider a long position in a forward contract.
  - An increase in market price reduces losses due to market but increase potential losses due to credit
  - A decrease in market price produces market losses but reduces credit exposure

The converse applies for a short forward position.

*Conclusion:* Market and Credit VARs offset
CVAR IV: Numerical Example

1. Per the numerical example, we seek

\[ \text{Var} (\Delta V - \Delta P_0) = \text{Var} (\Delta V) + \text{Var} (\Delta P_0) - 2 \text{Cov} (\Delta V, \Delta P_0) \]

2. From CVAR III,

\[ \text{Var} (\Delta V_t) = (0.368)^2 \]

3. Further

\[ \text{Var} (\Delta P_0) = (53 \times 0.2457)^2 \]

4. We now require

\[
\text{Cov} (\Delta V, \Delta P_0) = \text{Cov} \left( P_0 \frac{\partial V_t}{\partial P_0} \frac{dP_0}{P_0} + \frac{\partial V_t}{\partial p_t} p_t \frac{dp_t}{p_t}, P_0 \frac{dP_0}{P_0} \right)
\]
\[ = P_0^2 \frac{\partial V_t}{\partial P_0} \sigma_{P_0}^2 + \frac{\partial V_t}{\partial p_t} p_t P_0 \rho \sigma_p \sigma_{P_0} \]
\[ = 53^2 \times 0.2457^2 \times 0.0243 + 3.445 \times 0.075 \times 53 \times 0.6 \times 0.3 \times 0.2457 = 4.726 \]

5. Finally,

\[ \text{Var} (\Delta V - \Delta P_0) = (0.368)^2 + (53 \times 0.2457)^2 - 2 \times 4.726 = (12.66)^2 \]

\[ \implies \text{CVAR IV} = \frac{1.645 \sqrt{\text{Var} (\Delta V - \Delta P_0)} \Delta t}{\sqrt{52}} = 2.888 \]
SUMMARY

• CVAR implementation can be done analytically

• CVARs can be computed for
  – Credit Exposure at Risk (CVAR I)
  – Credit Risk Insurance (CVAR II)
  – Value-at-Risk computation (CVAR III) as well as VAR incorporating both market and credit risks (CVAR IV)
Appendix —

CVAR II: Estimating the Default Probability Process

Indirect Estimates of Default-Rate Volatilities

This approach combines the use of year-by-year marginal default probabilities $p_t$ together with transition-matrix values to determine the default volatilities $\sigma_{pt}$.

<table>
<thead>
<tr>
<th>Average Cumulative Default Rates, Year 1 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
</tr>
<tr>
<td>0.00</td>
</tr>
</tbody>
</table>

One-Year Transition Probabilities for a BBB-Rated Borrower

<table>
<thead>
<tr>
<th>Rating</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.02%</td>
</tr>
<tr>
<td>AA</td>
<td>0.33%</td>
</tr>
<tr>
<td>A</td>
<td>5.95%</td>
</tr>
<tr>
<td>BBB</td>
<td>86.93%</td>
</tr>
<tr>
<td>BB</td>
<td>5.3%</td>
</tr>
<tr>
<td>B</td>
<td>1.17%</td>
</tr>
<tr>
<td>CCC / Default</td>
<td>0.12% + 0.18% = 0.3%</td>
</tr>
</tbody>
</table>

Notationally, let each probability in the above table be denoted $q_i$, $i = 1, \ldots, 7$, and let each entry in the previous table be denoted $p_{1i}$. That is,

$$[p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}, p_{17}] = [0.00, 0.00, 0.06, 0.18, 1.06, 5.2, 19.79]$$

Then the calculation of $\bar{p}_1 \equiv \mathbb{E}(p_1)$, $\text{Var}(p_1)$ and $\text{Var}(\ln p_1)$ for each month in year 1 is given by:

$$\bar{p}_1 \equiv \mathbb{E}(p_1) = \sum_{i=1}^{7} q_i \frac{p_{1i}}{12}$$

$$\text{Var}(p_1) = \sum_{i=1}^{7} q_i \left( \frac{p_{1i}}{12} - \bar{p}_1 \right)^2 / 144$$

$$\text{Var}(\ln p_1) \approx \frac{\text{Var}(p_1)}{\bar{p}_1} \equiv \sigma_{p_1}^2$$