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Abstract: Facility location decisions play a critical role in transportation planning. In fact, it has recently become essential to study how such commitment integrate with the delivery of goods on an underlying transportation network problems especially when there are reasons to be uncertain about what are going to be those needs. In this research, we study a capacitated fixed-charge location-transportation problem in which, while the location and capacity of each facility need to be determined immediately, the determination of final production and distribution of products can be delayed until actual orders are received. In contexts where little is known about future demand, robust optimization, namely using a budgeted uncertainty set, becomes a natural method to employ in order to identify meaningful decisions. Unfortunately, it is well known that these types of two-stage robust problems are computationally intractable. To overcome this difficulty, we propose a set of conservative tractable approximations to the problem that each exploits to a different extent the idea of reducing the flexibility of the delayed decisions. While all of these approximations will outperform previous approximation models that have been proposed for this problem, each of them will have the potential to reach a different level of compromise between efficiency of resolution and quality of the solution. We will also demonstrate that full flexibility is often unnecessary to reach nearly, or even exact, optimal robust locations and capacities for the facilities. We illustrate our findings with an extensive numerical study where we evaluate the effect of the amount of uncertainty on the performance and structure of each approximate solutions that can be obtained.

Key Words: Transportation, facility location, robust optimization, flexibility, tractable approximation, demand uncertainty.

1 Introduction

Transportation planning can be decomposed in three different levels (Crainic and Laporte, 1997): strategic transportation planning, tactical transportation planning, operational transportation planning. At the highest level of management, an important decision consists in determining the geographical locations of factories, suppliers and warehouses. Determination of facility location, such as hub locations, supplier locations, air freight hub locations, railway station locations, etc., can significantly impact the design of the strategic networks. Recognizing this fact, researchers (e.g. Christensen et al. (2013), Abouee-Mehrizi et al. (2014)) have been developing integrated models in order to have better control on the interactions between facility location decisions and transportation strategies.

The traditional way of describing the location-transportation problem (LTP) has been to assume a deterministic environment. In a deterministic setting, i.e. when there is no uncertainty about problem data, the capacitated fixed-charge LTP can take the form of the following mixed integer linear program (MILP):

$$\text{(Deterministic model)}\ \underset{I,Z,Y,P}{\text{maximize}} \quad \sum_i \sum_j (\eta - d_{ij})Y_{ij} - c^T P - (C^T Z + K^T I) \tag{1a}$$

$$\text{subject to} \quad \sum_i Y_{ij} \leq D_j,\ \forall j, \tag{1b}$$

$$\sum_j Y_{ij} \leq P_i\ ,\ \forall i, \tag{1c}$$

$$P \leq Z,\ Z \leq MI, \tag{1d}$$

$$Y \in \mathbb{R}_+^{L\times N},\ I \in \{0,1\}^L \tag{1e}$$

with M chosen large enough, which integrates the optimization of both “strategic” and “operational” decisions. At the strategic level, it includes for each candidate location $i = 1, 2, \ldots, L$, the binary decision I_i denoting whether a facility should be opened or not, and the continuous decision Z_i denoting the production capacity of the facility. Once these are decided, operational decisions include P_i denoting how many goods is produced at each facility and Y_{ij} denoting how much is shipped from facility i to customers at location j. The demand at location $j = 1, 2, \ldots, N$ is characterized by D_j. The total profit generated by the company is computed based on sales revenue, with $\eta > 0$ the unit price of goods, on construction costs, composed for a given facility size Z_i of a fixed cost K_i and variable costs C_iZ_i, on production costs c_i, and finally on transportation costs, with d_{ij} being the unit cost for any shipment from location i to j.

In model (1), all parameters are considered to be known exactly at the time of making the decision. In practice however, some parameters, in particular the exact size of each demand D_j, is rarely known at the moment of building the facilities.[1] In recent years, studies made in a number of field of applications (Bertsimas et al. (2011), Gabrel et al. (2014b)) have demonstrated the effectiveness of robust optimization (RO) for handling uncertainty especially in cases where there is no valid argument to justify the choice of a distribution model. A naive application of robust optimization to LTP under demand uncertainty might lead to the following robust counterpart (RC)

$$\text{(RC)}\quad \underset{I,Z,Y,P}{\text{maximize}} \quad \sum_i \sum_j (\eta - d_{ij})Y_{ij} - c^T P - (C^T Z + K^T I) \tag{2a}$$

$$\text{subject to} \quad \sum_i Y_{ij} \leq D_j\ ,\ \forall j,\ \forall D \in \mathcal{D} \tag{2b}$$

$$\sum_j Y_{ij} \leq P_i\ ,\ \forall i, \tag{2c}$$

$$P \leq Z,\ Z \leq MI \tag{2d}$$

$$Y \in \mathbb{R}_+^{L\times N}\ , I_i \in \{0,1\}^L. \tag{2e}$$

where $\mathcal{D}$ is the uncertainty set for the vector of demand.

Although it can be shown that the RC model can be reformulated as a mixed integer linear program, the solution that it provides will often appear to be overly conservative: i.e. it might suggest to open only a few facilities (if any at all) with very limited capacity. This is actually due to the fact that the RC model completely disregards how operational decision, namely size of production and deliveries, are delayed and flexible so that they can exploit the information that becomes available about demand. This motivates the use of the following two-stage robust location-transportation problem (TRLTP) model :

$$\text{(TRLTP)} \quad \underset{I,Z}{\text{maximize}} \quad -(C^T Z + K^T I) + \min_{D \in \mathcal{D}} \; h(I, Z, D) \tag{3a}$$

$$\text{subject to} \quad Z \leq MI, \; I \in \{0,1\}^L. \tag{3b}$$

where $h(I, Z, D)$ is the achieved second-stage profit defined as

$$h(I, Z, D) = \underset{Y,P}{\text{maximize}} \quad \sum_i \sum_j (\eta - d_{ij}) Y_{ij} - c^T P \tag{4a}$$

$$\text{subject to} \quad \sum_i Y_{ij} \leq D_j \,, \forall j, \tag{4b}$$

$$\sum_j Y_{ij} \leq P_i \,, \forall i, \tag{4c}$$

$$P \leq Z \tag{4d}$$

$$Y \in \mathbb{R}_+^{L \times N}. \tag{4e}$$

Finally, we make the common assumption that the demand vector D is known to lie in a budgeted uncertainty set (see Bertsimas and Sim (2004)), i.e. that each D_i lies in an interval and that at most Γ of them can take extreme values.

While it appears that the TRLTP does implement as much flexibility as is needed in this problem, Atamtürk and Zhang (2007) established that evaluating the objective is already computationally intractable. In this paper, we present a set of five conservative approximation models to the problem that each exploits to a different extent the idea of reducing the flexibility of the delayed decisions. These models will allow us to explore empirically the compromises that need to be made between flexibility/conservatism and "tractability".[2] A side product of this study is to demonstrate that full flexibility is often unnecessary to reach nearly, or even exact, optimal robust locations and capacities for the facilities. We illustrate our findings with an extensive numerical study where we evaluate the effect of the amount of uncertainty on the performance and structure of each approximate solutions that can be obtained.

The remainder of the paper is organized as follows. In Section 2, we review prior work about the robust location transportation problem under demand uncertainty. In Section 3, we present our five tractable approximation models for the TRLTP (3). In Section 4, we establish the relation between the optimality gap of each approximation model. We next identify conditions under which some of the models return exact solutions. In Section 5, we provide numerical results and finally, the conclusions and possible future research directions are presented in Section 6.

2 Prior work

To the best of our knowledge, Atamtürk and Zhang (2007) were the first to study a model related to the TRLTP for an application of network flow and design problem where their objective was to minimize worst-case cost over a budgeted uncertainty set. They compared two-stage robust optimization model with stochastic programming where the objective of the stochastic program was to minimize the sum of the first-stage cost and expected value of the second-stage cost. When distribution was captured by 200 demand scenarios, they showed that while the solution of the two-stage robust optimization model increased the expected cost by 1.1% (from 1795 to 1815) it actually decreased by 29.1% (from 2869 to 2033) the cost incurred under the worst scenario. They identified the TRLTP as a special case of the modelling framework and after recognizing that the problem was NP-hard, proposed to use a cutting plane algorithm to reach a global optimum.

Recently Gabrel et al. (2014a) and Zeng and Zhao (2013) proposed two cutting plane methods to solve TRLTP exactly under the budgeted uncertainty set with an integer budget. Gabrel et al. (2014a) showed that the adversarial problem in the TRLTP could be reformulated as a MILP. The master problem of TRLTP could then be tackled using Kelley's cutting plane algorithm given that optimality cuts are provided using a MILP solver. Zeng and Zhao (2013) seem to have improved on the solution time by employing a column-and-constraint generation algorithm instead of Kelley's cutting plane algorithm. Although there is empirical evidence that both of these methods are efficient, the adversarial problem that is solved as a MILP is inherently NP-hard thus could lead to unbearable computation times for some instances.

In Baron et al. (2011), the authors can be considered to have proposed the first tractable approximation of the TRLTP model. In their paper, the authors proposed a model in which a fractional transportation policy is optimized. Indeed, they replaced the Y_{ij} variables with $X_{ij}D_j$ which reflects the notion that X_{ij} is the proportion of demand j satisfied by location i, namely that $0 \leq X_{ij}$ and $\sum_i X_{ij} \leq 1 \,\forall\, j$. Specifically, their proposed fractional variable-based (FVB) model takes the form:

$$\text{(FVB)} \quad \underset{I,Z,X,P}{\text{maximize}} \quad \min_{D\in\mathcal{D}} \sum_i \sum_j (\eta - d_{ij}) D_j X_{ij} - c^T P - (C^T Z + K^T I) \tag{5a}$$

$$\text{subject to} \quad 1^T X \leq 1, \tag{5b}$$

$$\sum_j D_j X_{ij} \leq P_i \;, \forall\, i, \; \forall D \in \mathcal{D}, \tag{5c}$$

$$P \leq Z, \; Z \leq MI \tag{5d}$$

$$X \in \mathbb{R}_+^{L\times N} \;, I_i \in \{0,1\}^L. \tag{5e}$$

They next studied the impact of two types of uncertainty set, box and ellipsoidal sets, on the structure of the robust solution and compared it to the nominal one. In particular, they paid special attention to the number of opened facilities, the total capacity of facilities, the number of deliveries made from each facility to the customer locations under different scenarios. Surprisingly, the following example highlights the fact that the solution of the FVB model might abandon opportunities of making profits that are arbitrarily large even with respect to the worst-case scenario. In contrast, the simpler RC model actually does not suggest as much of a conservative solution for the same instances.

Example 1 Consider an example of TRLTP with two customers such that $D \in [\bar{D} \pm \hat{D}]$ where $\bar{D} = 10000$ and $\hat{D} = 5000$. The location of customers is considered as the candidate location of facilities, $L = 2$. The open facility will cover demand, if possible, with $\eta = 1$, $c_i = 0.1$, $C_i = 0.1$ and $K_i = 3000$ for all i and the distance between locations is equal to 1. We assume that the budget is $\Gamma = 2$ which leads to a box uncertainty set. As it is shown in Appendix A, the optimal value of RC model (2) is equal to 2000 but the optimal value of the FVB model is zero in this example. This indicates that while the RC model suggests opening the two facilities which leads to a worst-case profit of 2000, the FVB model closes everything down. When scaling every parameters in the objective function by some $\alpha > 0$, FVB will let go of an arbitrarily large opportunity to make profit.

3 Tractable approximations

In what follows, we provide five ways of improving the quality of the solution obtained from the RC and FVB model. Each of them will employ the ideas of affine adjustments from Ben-Tal et al. (2004) and splitting based uncertainty set extensions from Chen and Zhang (2009) to exploit to a different extent the fact that the operational decisions P and Y can be adjusted to the realization of the demand. The type of flexibility added by our models can be divided in two classes. Similarly to what is done in the FVB model, the first class of approximation models, called "customer-driven", will adjust the size of a delivery to a customer simply based on information about that customer's demand, i.e. that $Y_{ij} := \pi_{ij}(D_j)$ with $\pi_{ij} : \mathbb{R} \to \mathbb{R}$. In opposition, the second class of approximation models, called "market-driven" will be more flexible and attempt to optimize delivery policies that take into account the state of the market as a whole, i.e. that $Y_{ij} := \pi_{ij}(D)$ with $\pi_{ij} : \mathbb{R}^N \to \mathbb{R}$. This second class will necessarily lead to models that are more computationally demanding. Note that in presenting each of the approximation models we omit to derive and spell out the finite dimensional MILP reformulation that would be obtained by applying duality theory to each robust constraint and objective function for the sake of keeping the presentation compact. In fact, our numerical study will later make use of the ROME package for Matlab (see Goh and Sim (2011)) in order to implement and solve each approximation model.

3.1 Customer-driven affine adjustments

Our first approximation model will stem from the realization that in the recourse problem of the TRLTP, namely (4), the inequality constraint (4c) will be active at optimum and can therefore be replaced with an equality constraint. This argument motivates replacing P_i with $\sum_j D_j X_{ij}$ in the FVB model (5). In order to ensure that we obtain a tighter approximation than with the RC model, we also propose replacing the fraction variable decision rule, $Y_{ij} := X_{ij} D_j$ by $Y_{ij} := X_{ij} D_j + W_{ij}$ which leads to our revised fractional-variable based (RFVB1) model:

$$\text{(RFVB1)} \quad \underset{I,Z,X,W}{\text{maximize}} \quad \min_{D \in \mathcal{D}} \sum_i \sum_j (\eta - d_{ij} - c_i)(X_{ij} D_j + W_{ij}) - (C^T Z + K^T I) \tag{6a}$$

$$\text{subject to} \quad \sum_i (X_{ij} D_j + W_{ij}) \leq D_j \,,\ \forall j,\ \forall D \in \mathcal{D} \tag{6b}$$

$$\sum_j (D_j X_{ij} + W_{ij}) \leq Z_i \,,\ \forall i,\ \forall D \in \mathcal{D} \tag{6c}$$

$$X_{ij} D_j + W_{ij} \geq 0 \,,\ \forall i, \forall j,\ \forall D \in \mathcal{D} \tag{6d}$$

$$Z \leq M I_i,\ I \in \{0,1\}^L. \tag{6e}$$

We next will exploit an extended description of the budgeted uncertainty set proposed in Chen and Zhang (2009) in order to optimize customer-driven transportation policies that have piecewise-linear structure. Specifically, we employ a lifting of the demand uncertainty space

$$\mathcal{D} = \left\{ D \in \mathbb{R}^N \middle| \exists (D^+, D^-) \in \mathcal{D}_2,\ D = \bar{D} + D^+ - D^- \right\}$$

where

$$\mathcal{D}_2 = \left\{ (D^+, D^-) \middle| \begin{array}{c} \exists (\delta^+, \delta^-) \in \mathbb{R}^N \times \mathbb{R}^N,\ \delta^+ \geq 0,\ \delta^- \geq 0, \|\delta^+ + \delta^-\|_\infty \leq 1, \\ \|\delta^+ + \delta^-\|_1 \leq \Gamma,\ D^+ = \delta^+ \hat{D},\ D^- = \delta^- \hat{D} \end{array} \right\}.$$

Intuitively, this lifting allows one to define different affine policies for positive perturbations than those defined for negative perturbations. For example, this gives the possibility to make a better use of small facilities by considering policies that serve a fix number of customers when demand is above its expected amount while serving a larger proportion of it when it is smaller than expected. This manipulation of the model leads to our second revision of the fractional-variable based (RFVB2) model:

$$\text{(RFVB2 model)} \quad \underset{I,Z,X,W}{\text{maximize}} \quad \min_{(D^+,D^-) \in \mathcal{D}_2} \sum_i \sum_j (\eta - d_{ij} - c_i)(X_{ij} D_j^+ + X_{ij}^- D_j^- + W_{ij}) - (C^T Z + K^T I) \tag{7a}$$

$$\text{subject to} \quad \sum_i (X_{ij} D_j^+ + X_{ij}^- D_j^- + W_{ij}) \leq D_j \,,\ \forall j,\ \forall\, (D^+, D^-) \in \mathcal{D}_2 \tag{7b}$$

$$\sum_j (X_{ij} D_j^+ + X_{ij}^- D_j^- + W_{ij}) \leq Z_i \,,\ \forall i,\ \forall\, (D^+, D^-) \in \mathcal{D}_2 \tag{7c}$$

$$X_{ij} D_j^+ + X_{ij}^- D_j^- + W_{ij} \geq 0 \,,\ \forall i, \forall j,\ \forall\, (D^+, D^-) \in \mathcal{D}_2 \tag{7d}$$

$$Z \leq M I_i,\ I \in \{0,1\}^L. \tag{7e}$$

3.2 Market-driven affine adjustments

We now provide three approximation models that will attempt to exploit full market information in making deliveries. The first of these attempts can be considered as a direct application of the affinely adjustable robust counterpart (AARC) framework for the TRLTP as it was initially introduced by Ben-Tal et al. (2004). In such a framework, the adaptive policies for second-stage decisions are considered to be restricted to the set of affine functions of the uncertain parameters. In the context of this problem, this means that each adaptive policy for $Y_{ij} : \mathbb{R}^N \to \mathbb{R}$ of the TRLP model (3) should take the form $Y_{ij} := X_{ij}^T D + W_{ij}$ with $X_{ij} \in \mathbb{R}^N$ and $W_{ij} \in \mathbb{R}$. In other words, this means that the delivery for a customer j will depend on all the orders that are made in this market. We note that similarly as before the variable P_i of the TRLPT model (3) will be replaced by $\sum_j X_{ij} D_j + W_{ij}$ in all of our proposed approximations. When restricting our search to affine policies of the D vector, the approximation model takes the following form:

$$\begin{aligned}
\text{(AARC)} \quad \underset{I,Z,X,W}{\text{maximize}} \quad & \min_{D \in \mathcal{D}} \sum_i \sum_j (\eta - d_{ij} - c_i)(X_{ij}^T D + W_{ij}) - (C^T Z + K^T I) \\
\text{subject to} \quad & \sum_i (X_{ij}^T D + W_{ij}) \leq D_j \,, \forall j, \; \forall \, D \in \mathcal{D} \\
& \sum_j (X_{ij}^T D + W_{ij}) \leq Z_i \,, \forall i, \; \forall \, D \in \mathcal{D} \\
& X_{ij}^T D + W_{ij} \geq 0 \,, \forall i, \forall j, \; \forall \, D \in \mathcal{D} \\
& Z \leq MI, I \in \{0,1\}^L.
\end{aligned} \tag{8}$$

Similar to what was done to obtain the RFVB2 model, AARC can be improved by lifting the uncertainty set. The lifted affinely adjustable robust counterpart (LAARC) of TRLTP (3) can be obtained by considering policies that are affine in the pair of perturbations $(D^+, D^-) \in \mathcal{D}_2$, namely $Y_{ij} := {X_{ij}^+}^T D^+ + {X_{ij}^-}^T D^- + W_{ij}$ with $X_{ij}^+ \in \mathbb{R}^N$, $X_{ij}^- \in \mathbb{R}^N$, and $W_{ij} \in \mathbb{R}$. This new approximation model takes the more sophisticated form:

(LAARC)

$$\underset{I,Z,X^+,X^-,W}{\text{maximize}} \quad \min_{(D^+,D^-) \in \mathcal{D}_2} \sum_i \sum_j (\eta - d_{ij} - c_i)({X_{ij}^+}^T D^+ + {X_{ij}^-}^T D^- + W_{ij}) - (C^T Z + K^T I) \tag{9a}$$

$$\text{subject to} \quad \sum_i ({X_{ij}^+}^T D^+ + {X_{ij}^-}^T D^- + W_{ij}) \leq D_j \,, \forall j, \; \forall \, (D^+, D^-) \in \mathcal{D}_2 \tag{9b}$$

$$\sum_j ({X_{ij}^+}^T D^+ + {X_{ij}^-}^T D^- + W_{ij}) \leq Z_i \,, \forall i, \; \forall \, (D^+, D^-) \in \mathcal{D}_2 \tag{9c}$$

$${X_{ij}^+}^T D^+ + {X_{ij}^-}^T D^- + W_{ij} \geq 0 \,, \forall i, \forall j, \; \forall \, (D^+, D^-) \in \mathcal{D}_2 \tag{9d}$$

$$Z \leq MI, \; I \in \{0,1\}^L. \tag{9e}$$

Now, we propose an extension to the LAARC, referred as the ELAARC model, which will benefit from a manipulation of a two-stage robust optimization model which to the best of our knowledge is presented here for the first time. The key idea is to reformulate the recourse problem(4) in a way that relaxes the constraint that is plagued by uncertainty without compromising the authenticity of the model. Namely, let us consider the following equivalent reformulation

$$h(I, Z, D) = \underset{Y,P}{\text{maximize}} \quad \sum_i \sum_j (\eta - d_{ij} - c_i) Y_{ij} - \sum_j B_j \theta_j \tag{10a}$$

$$\text{subject to} \quad \sum_i Y_{ij} \leq D_j + \theta_j \,, \forall j, \tag{10b}$$

$$\sum_j Y_{ij} \leq Z_i \,, \forall i, \tag{10c}$$

$$Y \in \mathbb{R}_+^{L \times N}, \; \theta \in \mathbb{R}_+^N. \tag{10d}$$

where $B_j \in \mathbb{R}$ is a marginal penalty for violating constraint (4b) that is chosen large enough for the optimal value of the optimization problem to remain the same. We refer the reader to Appendix B for a proof that the assignment $B_j = \max_i(\eta - c_i - d_{ij}) \;\; \forall j$ meets this criterion.

Similar as for the LAARC model, we adjust the deliveries based on the lifted uncertainty space, $Y_{ij} := {X_{ij}^{+}}^T D^+ + {X_{ij}^{-}}^T D^- + W_{ij}$, furthermore we adjust each new auxiliary variable θ_j according to $\theta_j := S_j^+ D_j^+ + S_j^- D_j^-$ in order to obtain the ELAARC approximation model

$$
\begin{aligned}
\text{(ELAARC)} \quad \underset{\substack{I,Z,X^+,X^-,\\ W,S^+,S^-}}{\text{maximize}} \quad & \min_{(D^+,D^-)\in\mathcal{D}_2} \sum_i \sum_j (\eta - d_{ij} - c_i)({X_{ij}^{+}}^T D^+ + {X_{ij}^{-}}^T D^- + W_{ij}) \\
& \qquad -(C^T Z + K^T I) - \sum_j B_j (S_j^+ D_j^+ + S_j^- D_j^-) && \text{(11a)} \\
\text{subject to} \quad & \sum_i ({X_{ij}^{+}}^T D^+ + {X_{ij}^{-}}^T D^- + W_{ij}) \le D_j \\
& \qquad + S_j^+ D_j^+ + S_j^- D_j^- \,, \forall j, \; \forall\, (D^+, D^-) \in \mathcal{D}_2 && \text{(11b)} \\
& \sum_j ({X_{ij}^{+}}^T D^+ + {X_{ij}^{-}}^T D^- + W_{ij}) \le Z_i \,, \forall i, \; \forall\, (D^+, D^-) \in \mathcal{D}_2 && \text{(11c)} \\
& {X_{ij}^{+}}^T D^+ + {X_{ij}^{-}}^T D^- + W_{ij} \ge 0 \,, \forall i, \forall j, \; \forall\, (D^+, D^-) \in \mathcal{D}_2 && \text{(11d)} \\
& S_j^+ D_j^+ + S_j^- D_j^- \ge 0 \; \forall j, \; \forall\, (D^+, D^-) \in \mathcal{D}_2 && \text{(11e)} \\
& Z \le MI, \; I \in \{0,1\}^L. && \text{(11f)}
\end{aligned}
$$

4 Theoretical analysis of robust approximation models

In this section, we are interested in demonstrating theoretically how solutions of better quality can be obtained by using an approximation model that offers more flexibility for the delayed decisions. In particular, we start by establishing what are the respective qualities of the bounds that are obtained from each model regarding the worst-case profit of a candidate solution for facility locations and capacities.

Proposition 1 *Given some values for the strategic decision vectors $I \in \{0,1\}^L$ and $Z \in \mathbb{R}^L$, let $f_{\text{TRLTP}}(I,Z)$, $f_{\text{FVB}}(I,Z)$, $f_{\text{RFVB1}}(I,Z)$, $f_{\text{RFVB2}}(I,Z)$, $f_{\text{AARC}}(I,Z)$, $f_{\text{LAARC}}(I,Z)$, and $f_{\text{ELAARC}}(I,Z)$ respectively be the value of the objective functions of approximation models (3), (5), (6), (7), (8), (9), and (11). The following partial ordering is satisfied for any values of I and Z:*

$$f_{\text{FVB}}(I,Z) \le f_{\text{RFVB1}}(I,Z) \le f_{\text{RFVB2}}(I,Z) \le f_{\text{LAARC}}(I,Z) \le f_{\text{ELAARC}}(I,Z) \le f_{\text{TRLTP}}(I,Z),$$
$$f_{\text{RFVB1}}(I,Z) \le f_{\text{AARC}}(I,Z) \le f_{\text{LAARC}}(I,Z)$$

Proof. The ELAARC is a lower bound on true worst-case profit $f_{\text{TRLTP}}(I,Z)$ since adjustable variables that appear in problem (10) are limited to affine function of uncertain parameter. The ELAARC model reduces to the LAARC model when the value of variables S_j^+ and S_j^- are equal to zero for all j. One can also show that the LAARC model reduces to the AARC model when one adds the constraint that the policy must additionally be affine in D, i.e. that $X_{ij}^+ = -X_{ij}^- \,\forall i, j$, thus leading to a lower evaluation of worst-case second-stage profit. The LAARC model also reduces to the RFVB2 model when adding the constraints that X_{ij} equals zero except for the j-th term. A similar set of constraints make the AARC model reduce to RFVB1 model. The RFVB2 model reduces to the RFVB1 model in a similar way that LAARC reduces to AARC.

Finally, the evaluation of worst-case profit obtained from the RFVB1 model, even with $W = 0$, can be shown to upper bound the evaluation from the FVB model since one can replace constraint (6c) with

$$\sum_j D_j X_{ij} \le P_i \,, \forall i, \; \forall D \in \mathcal{D} \;\; \& \;\; P_i \le Z_i \,\forall i \,,$$

after letting P_i become an additional decision variable of the model and since the objective function of the RFVB1 model has the following properties

$$\min_{D\in\mathcal{D}} \sum_i \sum_j (\eta - d_{ij} - c_i) D_j X_{ij} - (C^T Z + K^T I) = \sum_i \sum_j (\eta - d_{ij} - c_i) D_j X_{ij} - (C^T Z + K^T I) - cP + cP$$

$$
\begin{aligned}
&= \min_{D\in\mathcal{D}} \sum_i \sum_j (\eta - d_{ij}) D_j X_{ij} - (C^T Z + K^T I) - cP + \sum_i c_i (P_i - D_j X_{ij}) \\
&\geq \min_{D\in\mathcal{D}} \sum_i \sum_j (\eta - d_{ij}) D_j X_{ij} - (C^T Z + K^T I) - cP\,.
\end{aligned}
$$

In this derivation, the last inequality comes from the robust constraint $\sum_j D_j X_{ij} \leq P_i \; \forall D \in \mathcal{D}$. Since this last expression is the objective function of the FVB model, it is clear that the optimal value of this problem will be lower than the value of the RFVB1 model. □

The result presented in Proposition (1) can easily be used to establish guarantees with respect to the optimal worst-case profits that are evaluated by each model.

Corollary 1 *Let $f^*_{TRLTP}, f^*_{FVB}, f^*_{RFVB1}, f^*_{RFVB1}, f^*_{AARC}, f^*_{LAARC}$ and f^*_{ELAARC} respectively be the optimal value of (3), (5), (6), (7), (8), (9), and (11). The following partial ordering is always satisfied:*

$$
f^*_{FVB} \leq f^*_{RFVB1} \leq f^*_{RFVB2} \leq f^*_{LAARC} \leq f^*_{ELAARC} \leq f^*_{TRLTP} \;\&\; f^*_{RFVB1} \leq f^*_{AARC} \leq f^*_{LAARC}\,.
$$

Together, these results show that more sophisticated models of this list always provide better conservative approximation of the optimal value TRLTP model (See Figure 1). In fact, anytime one approximation model in this list returns exactly the optimal value of the TRLTP, all models that are higher or equal to it in this ordering is guaranteed to return an exact optimal solution and exact optimal worst-case bound.

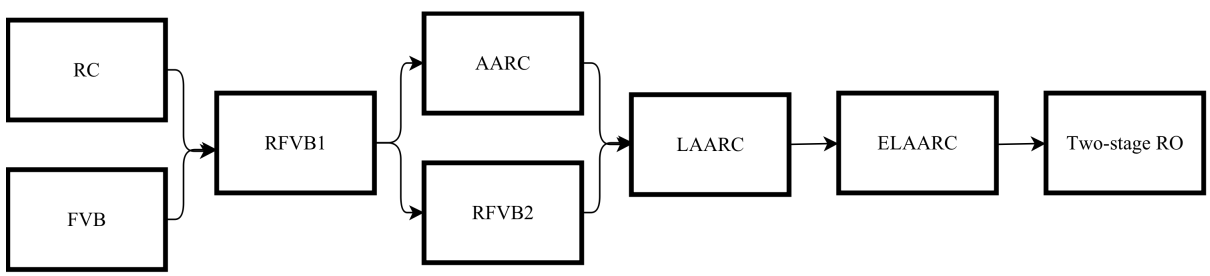


Figure 1: Partial ordering of the quality of bounds obtained from the different approximation models. Each arrow connects an approximation model to an approximation model that returns a tighter bound on the optimal worst-case profit of the TRLTP model.

In the following theorem, we present conditions under which the proposed approximations are optimal.

Theorem 1 *The TRLTP model (3) is equivalent to:*

- *RFVB1, RFVB2, AARC, LAARC and ELAARC when $C = 0$,*
- *RC, RFVB1, RFVB2, AARC, LAARC and ELAARC when $\Gamma = N$ (i.e. box set),*
- *LAARC and ELAARC when $\Gamma = 1$.*

Proof. Appendix C. □

Corollary 1 and Theorem 1 imply that LAARC and ELAARC not only provide tighter bound than all proposed approximation models but also are optimal for TRLTP for a number of interesting uncertainty sets.

5 Numerical results

In this section, we evaluate the proposed approximation models on a set of randomly generated problem instances using a method that was proposed in (Baron et al., 2011). We randomly generate N nodes on a unit square, representing the demand points and choose randomly L nodes of this N nodes as candidate facility locations. Each facility i is associated to the same parameters $K_i = 50000$, $C_i = 0.1$, $c_i = 0.1$, $\eta = 1$. On the other hand, the interval uncertainty for demand at each customer location j takes the shape $[(1-\varepsilon)\bar{D}_j, (1+\varepsilon)\bar{D}_j]$ with $\varepsilon = 0.15$ and $\bar{D}_j$ uniformly drawn between 17500 units and 22500 units. The respective unit transportation cost between a facility and a customer location, d_{ij}, is simply considered equal to the Euclidean distance between the two.

The topics we seek to address are:

- the computational time of each approximation models,
- the impact of different amount of uncertainty on the quality of the quality of the worst-case bound of each approximation model,
- some insights about the structure of the decisions that are proposed by each approximation model, namely in terms of number open facilities and total capacity of open facilities,
- statistics for each approximate solution proposed about the amount of demand that is covered and the amount of unused capacity under different scenarios,
- the potential of each model with respect to trading-off average performance and robustness.

5.1 Computational analysis

In this subsection, we compare the computational time of each approximation model for varying size of the budget of uncertainty. As it is reported in Table 1, we consider three different sizes of instances including small size instances with $L = 10$, $N = 10$, medium size instances with $L = 10$ and $N = 20$, and large size instances with $L = 50$ and $N = 100$. The first observation is that the customer-driven models (i.e FVB, RFVB1 and RFVB2) benefit from strong computationally efficiency. It is observed that RFVB1 and RFVB2 model do not have significant difference in running time and although they increase the running time of the FVB model by an order of magnitude in large size instances, they can always be solved in less than two minutes. On the other hands, the running time of market-driven models (i.e. AARC, LAARC, and ELAARC) exceed to 12 minutes in medium size instances and they are not terminated within an hour for larger instances. The reason for this difference between customer-driven vs. market-driven models is the fact that the number of parameters that describe the policies are respectively of order $O(LN)$ and order $O(LN^2)$. Finally, it is observed that there is not a high computational price to pay to choose to migrate to the more complex models LAARC and ELAARC.

Table 1: Computational time of robust approximations (in seconds).

L	N	Γ(% of N)	FVB	RFVB1	RFVB2	AARC	LAARC	ELAARC
10	10	10	0.91	0.33	0.31	3.38	1.69	1.82
		30	0.28	0.40	0.35	4.87	2.81	3.91
		50	0.28	0.36	0.43	4.47	2.80	4.03
		70	0.26	0.36	0.32	4.06	3.10	3.89
		90	0.24	0.77	0.30	3.18	5.04	4.93
		Total	1.97	2.22	1.71	19.96	15.44	18.58
10	20	10	0.32	0.74	0.52	30.58	95.30	145.05
		30	0.32	0.73	0.50	35.00	92.89	102.15
		50	0.38	0.51	0.46	56.50	96.04	116.79
		70	0.29	0.47	0.49	60.52	102.19	132.83
		90	0.29	0.50	0.47	67.68	148.87	185.12
		Total	1.60	2.95	2.44	250.28	535.29	681.94
50	100	10	4.00	77.85	34.97	-	-	-
		30	4.95	49.31	42.82	-	-	-
		50	3.73	74.64	44.03	-	-	-
		70	3.40	20.35	27.89	-	-	-
		90	3.27	27.40	27.98	-	-	-
		Total	19.35	249.55	177.69	-	-	-

5.2 Impact of size of perturbation on optimality gap

We consider 100 problem instances with $L = 10, N = 10$ and evaluate, under different size of demand intervals and total budgets of uncertainty, the worst-case bound that is obtained by each approximation model. We also evaluated for each approximate solution what was the worst-case performance that would be achieved by implementing the strategic part of the decision, namely I and Z. Tables 2, 3 and 4 present for each approximation model an average, taken over 100 instances, of the relative difference between the actual optimal worst-case profit attainable in the TRLTP model and both the bound that is produced and the achieved worst-case performance.[3] One might first observe in these tables that as indicated by Corollary 1, the worst-case profit bounds are always improved by using a tighter approximation model. The most significant improvement appears to occur when passing from the FVB model to the RFVB1 model and later by passing to a market-driven model. For instance, in Table 4, when $\Gamma = 5$, the FVB model always wastes all possible profits while ELAARC provides a strategic decision that is on average 0.26% from being optimal and can guarantee a lower bound on profits that is on average 0.48% below the best worst-case profit. It can also be observed that all of our proposed methods are exact for the case with $\Gamma = N$ as stated in Theorem 1, moreover the LAARC and ELAARC models are optimal for $\Gamma = 1$ and appear to be always optimal, although not formally proved, when $\Gamma = N - 1$.

When we look at the results of FVB model and other customer-driven models, we observe that RFVB1 and RFVB2 models improve FVB model such that they reduce by factor 8 the quality of the worst-case bound offered by the FVB model. In particular, one might notice that when $\varepsilon = 0.30$, the company always identifies profitability in servicing its customers under the RFVB1 and RFVB2 models while the FVB model suggests shutting down the business at $\Gamma = 4$. This is somehow alarming evidence that the FVB model is overly conservative.

Furthermore, it appears that a significant gain is achieved with the introduction of market-driven policies such that the proposed approximate bounds are on average always less than 0.57% from being exact. Although the added value of using the LAARC and ELAARC models is not very pronounced (refer to underlined and bold entries respectively), the difference becomes more remarkable as the size of demand interval is increased.

It is also observed that the difference between worst-case bound and worst-case profit of the FVB model can be significant (e.g. when $\Gamma = 6$ and $\varepsilon = 0.15$ where the difference exceeds 37.6%). In comparison, the difference is much smaller for other models. Regarding sensitivity to the size of Γ and ε, one might notice that the quality of worst-case bounds decrease when more uncertainty is present yet the performance of the AARC, the LAARC, and the ELAARC models appear to be less sensitive.

Table 5 presents further statistics regarding the sub-optimality of the strategic decisions proposed by each method. Specifically, for all 100 instances with 10 different Γ and 3 different ε (3000 trials), the table indicates for what percentage of random trials, each method has been able to provide decisions for I and Z which worst-case performance $f(I, Z)$ was within a given range from being optimal. One can observe that ELAARC improves AARC and LAARC models such that it reduces the maximum optimality gap of these models by a factor of 2. It is also observed that RFVB1 and RFVB2 models significantly improve the FVB model such that the percentage of trials with 100% optimality gap is decreased from 65.9% to almost 1% using RFVB1 and RFVB2 models. Finally, all market-driven models provide extremely efficient decisions as portrayed by a 0.1% optimality gap in more than 91 percent of trials.

5.3 Robustness-performance trade-off

In this subsection, we study the robustness and performance of approximations. In these experiments, we generate one hundred random problem instances where each facility i is associated to the same parameters $c_i = 0.5$ while parameters η, C_i, K_i are randomly chosen from the interval $[1.5, 2]$, $[0.5, 0.1]$, $[0, 20000]$ respectively. The interval uncertainty for demand at each customer location j takes the shape $[\bar{D}_j - \hat{D}_j, \bar{D}_j + \hat{D}_j]$ with $\bar{D}_j$ uniformly drawn form $[0, 20000]$ units and $\hat{D}_j = \varepsilon_j \bar{D}_j$ where ε_j is chosen randomly between 0.15 and 1 for all i. We consider two size of instances, medium size instances with $L = 10$ and $N = 20$ and large size instances with $L = 50$ and $N = 100$. For each instance, we will evaluate the performance of approximate robust solutions on sets of 100 demand scenarios. In each of these scenario, each customer's demand is

Table 2: Average relative difference between actual TRLTP worst-case profit and both the approximate bound or the achieved worst-case profit for each approximation model when ε=0.15.

Γ	FVB		RFVB1		RFVB2		AARC		LAARC		ELAARC	
	Bound	WC	Bound	WC	Bound	WC	Bound	WC	Bound	WC	Bound	WC
1	31.0	11.9	11.3	4.88	5.47	2.79	0	0	0	0	0	0
2	52.7	25.5	11.4	4.30	7.32	3.48	0.02	0.01	0.02	0.01	0.02	0.01
3	66.5	36.2	8.55	4.44	7.02	2.39	0.04	0.04	0.04	0.04	**0.03**	0.03
4	76.4	42.7	5.84	3.39	5.61	2.48	0.05	0.04	0.05	0.04	**0.04**	0.04
5	83.2	47.2	3.75	2.35	3.75	2.35	0.04	0.03	0.04	0.03	0.04	0.03
6	87.5	49.9	2.15	1.49	2.15	1.49	0.01	0.01	0.01	0.01	0.01	0.01
7	90.0	54.1	1.08	0.83	1.08	0.83	0.01	0.01	0.01	0.01	0.01	0.01
8	91.2	53.9	0.41	0.35	0.41	0.35	0.01	0.01	0.01	0.01	0.01	0.01
9	91.5	55.5	0.09	0.08	0.09	0.08	0.00	0.00	0	0	0	0
10	91.5	55.2	0	0	0	0	0	0	0	0	0	0

Table 3: Average relative difference between actual TRLTP worst-case profit and both the approximate bound or the achieved worst-case profit for each approximation model when ε=0.30.

Γ	FVB		RFVB1		RFVB2		AARC		LAARC		ELAARC	
	Bound	WC	Bound	WC	Bound	WC	Bound	WC	Bound	WC	Bound	WC
1	55.0	26.2	22.6	10.2	11.3	6.05	0	0	0	0	0	0
2	82.4	47.7	24.4	10.5	15.5	7.55	0.06	0.05	0.06	0.05	**0.05**	0.04
3	95.4	74.5	19.2	11.7	15.7	6.14	0.12	0.09	0.12	0.09	**0.08**	0.08
4	99.7	95.4	13.6	8.96	13.1	6.69	0.15	0.11	0.15	0.11	**0.12**	0.11
5	100.0	100.0	9.07	6.43	9.07	6.45	0.15	0.10	0.15	0.10	**0.13**	0.09
6	100.0	100.0	5.31	4.01	5.31	4.01	0.11	0.08	0.10	0.07	**0.09**	0.06
7	100.0	100.0	2.68	2.14	2.68	2.14	0.05	0.05	0.04	0.04	0.04	0.04
8	100.0	100.0	1.02	0.90	1.02	0.90	0.03	0.03	0.02	0.02	0.02	0.02
9	100.0	100.0	0.22	0.22	0.22	0.22	0.00	0.00	0	0	0	0
10	100.0	100.0	0	0	0	0	0	0	0	0	0	0

Table 4: Average relative difference between actual TRLTP worst-case profit and both the approximate bound or the achieved worst-case profit for each approximation model when ε=0.45.

Γ	FVB		RFVB1		RFVB2		AARC		LAARC		ELAARC	
	Bound	WC	Bound	WC	Bound	WC	Bound	WC	Bound	WC	Bound	WC
1	70.7	34.7	34.8	16.5	17.6	9.34	0	0	0	0	0	0
2	95.5	75.0	39.7	20.4	25.5	14.3	0.20	0.14	0.20	0.14	**0.11**	0.09
3	100.0	98.9	32.6	22.2	26.9	10.7	0.40	0.30	0.40	0.30	**0.26**	0.21
4	100.0	100.0	23.9	17.6	23.2	14.0	0.57	0.41	0.57	0.41	**0.44**	0.37
5	100.0	100.0	16.2	12.4	16.2	12.3	0.56	0.34	0.52	0.31	**0.48**	0.26
6	100.0	100.0	9.67	7.65	9.67	7.65	0.46	0.25	0.37	0.23	**0.33**	0.21
7	100.0	100.0	4.84	4.03	4.84	4.03	0.31	0.27	0.16	0.14	**0.15**	0.13
8	100.0	100.0	1.80	1.57	1.80	1.57	0.11	0.06	0.03	0.03	0.03	0.03
9	100.0	100.0	0.36	0.36	0.36	0.36	0.01	0.01	0	0	0	0
10	100.0	100.0	0	0	0	0	0	0	0	0	0	0

Table 5: Proportion of random instances (in 3000 trials) where approximate strategic decisions were within a certain range of the optimal worst-case profit for different values of Γ and ε reported.

Gap range	FVB	RFVB1	RFVB2	AARC	LAARC	ELAARC
$= 0$	14.13	42.20	42.20	85.70	86.37	87.90
≤ 0.1	14.20	44.77	44.90	91.70	92.13	93.70
≤ 1	15.20	54.93	56.93	98.77	98.90	99.33
≤ 10	19.47	83.67	88.30	99.97	99.97	100
$= 100$	65.90	1.10	0.97	0	0	0
Max gap	100	100	100	12.68	12.68	6.30

independently generated from its respective demand interval using a uniform distribution. In the instances of larger size, due to the duration of the resolution process, we limit our study to FVB, RFVB1 and RFVB2 models.

In Figure 2, we report the average expected profit and the average 90 percentile profit of each approximation model's solution as total budget for the uncertainty set is varied. The same results are also presented in Figure 3 to highlight what type of compromise can be achieved by adjusting the budget of uncertainty. Considering that a common criticism of robust optimization approaches has been that it provides overly conservative solutions, it might come as a surprise that our results show that a flexible robust optimization approach with an appropriately calibrated uncertainty set (e.g. the LAARC model with $\Gamma = 1$) will provide solutions that outperform the solutions of the deterministic model (1), obtained by setting $\Gamma = 0$, in terms of both expected profit and risk exposure as measured through the 90th percentile. Another interesting observation is that overly conservative solutions might often actually be the result of not injecting enough flexibility in the robust optimization model, as is the case for the FVB and RFVB1 models. The figures clearly show

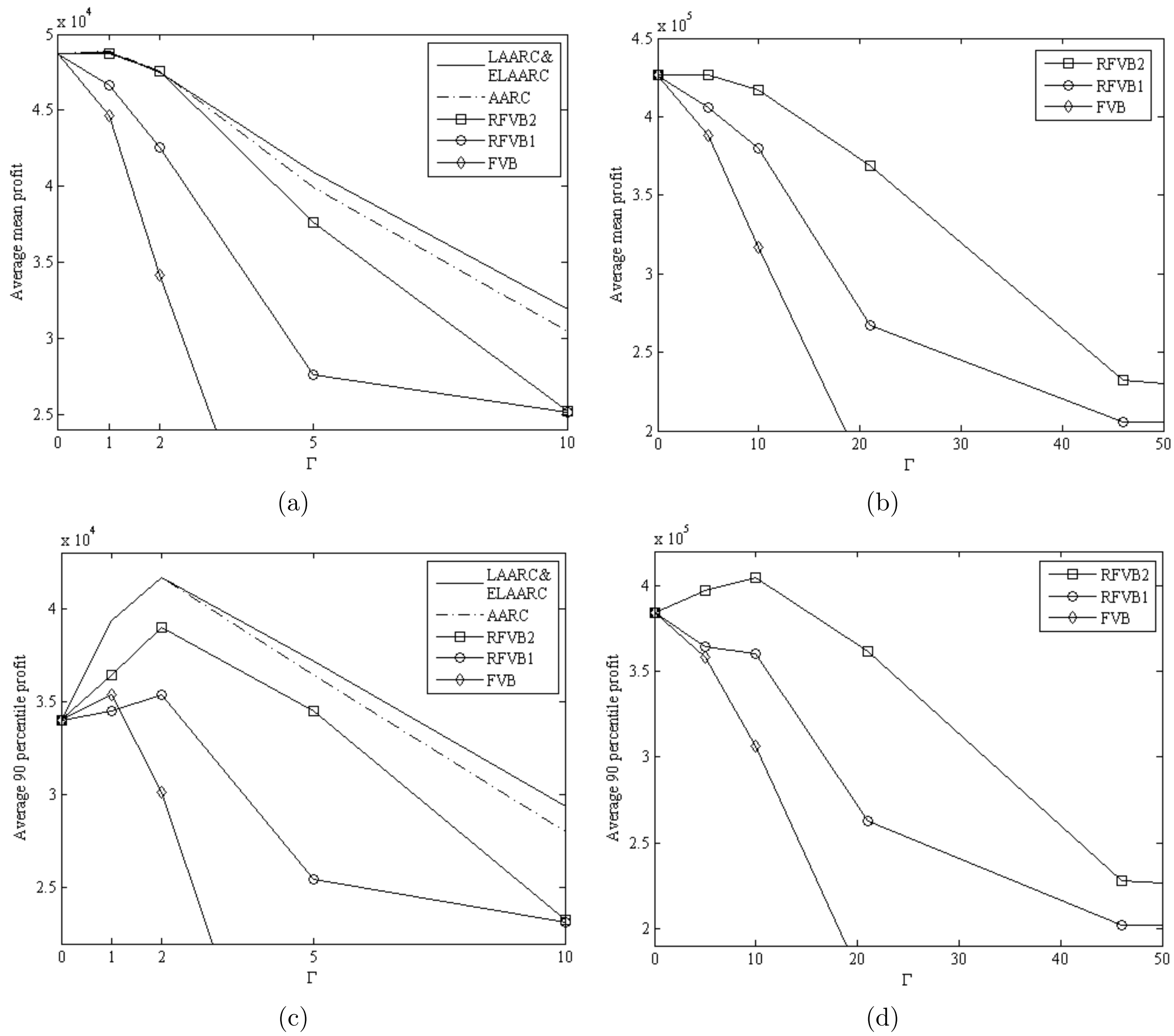
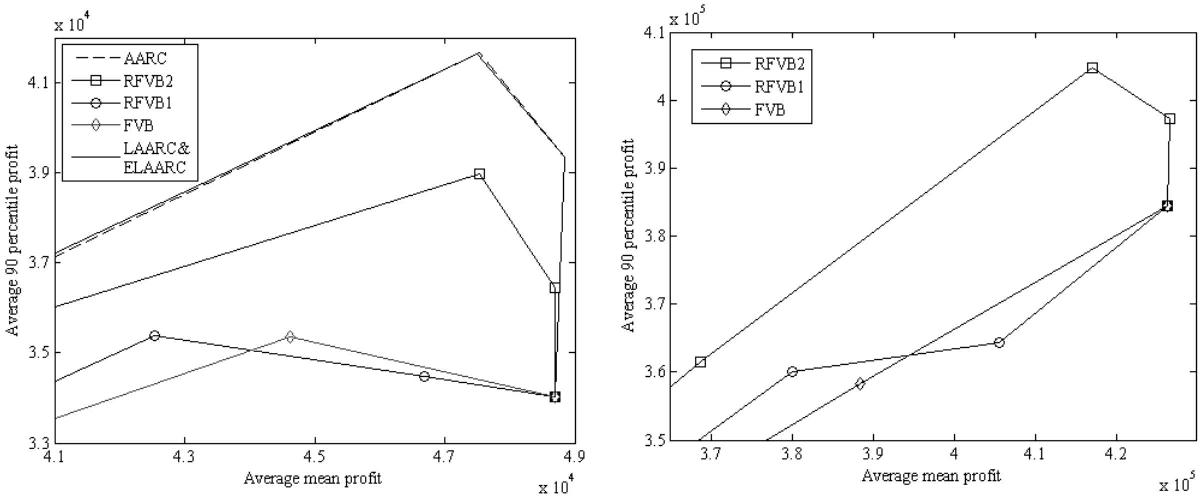


Figure 2: Average expected and 90th percentile profit achieved by the different robust methods on 100 problem instances when adjusting the level of conservativeness Γ. Figure (a) and (c) present the average expected and average 90th percentile profit respectively for medium sized instances with $L = 10$ and $N = 20$ while (b) and (d) present the same statistics but for for large sized instances with $L = 50$ and $N = 100$. Note that in (a) and (b) the curves for LAARC and ELAARC were combined since the performances where indistinguishable.



(a) (b)

Figure 3: Average expected profit versus average 90th percentile profit achieved by the different robust methods on 100 problem instances when adjusting the level of conservativeness Γ. Figure (a) presents the achieved risk-return trade-off for instances of medium size while (b) presents it for instances of large size. Note that in (a) the curves for LAARC and ELAARC were combined since the performances where indistinguishable.

that whether the instance is small or large, it is always worth employing the slightly more sophisticated RFVB2 model to achieve significantly better risk and return trade-off. Figure 2(a) also demonstrate how performance is improved by employing market-driven models which makes it somewhat unfortunate that it would come at such a high computational price as seen in Section 5.1.

5.4 Decision structure

In this subsection, we study the strategies that are obtained from our approximation models. In particular, we look at characteristics such as the number of facilities that are opened and the total production capacities that is installed. To perform this analysis, we replicate the experiments that were done in Section 5.3. Statistics of these experiments are reported in Table 6. In particular, its first set of rows indicates the proportion of problem instances where at least one facility location was proposed for different levels of uncertainty budgets. Once again, the over conservatism of the FVB model can be observed as the model refuses to open any facility in 43% of instances for a relatively small value of $\Gamma = 2$. In contrast, the proportion of problem instances where no facilities are selected is below 15% for all other approximation models. In the other two sets of rows of Table 6, we reports the number of open facilities and total capacity of the proposed solution averaged over the instances where at least one facility location was selected. Regarding the strategies proposed by each models, one might notice that more flexible models always propose opening a larger number of facilities. However, the same cannot be said of total capacity. In particular, it appears that when $\Gamma = 1$ market-driven models are a bit more cautious with respect to the capacity of its facilities. Increasing the amount of uncertainty has the natural effect of encouraging a smaller number of smaller facilities. It might also be worth underlining the fact that although the FVB model tends to propose the smallest number of facilities, it is mislead to promote much larger ones. We believe all these results reaffirm the added value that is obtained by including more flexible policies in the robust optimization model.

We conclude this numerical study with Table 7 which describes how much each approximation model is able to cover the realized demand and make efficient use of its capacity as the uncertainty budget Γ is increased. The first observation one can make is that the percentage of covered demand and the percentage of unused capacity displays increased caution, i.e. a decrease of both percentages, as the models account for increased uncertainty through Γ. We also observe that market-driven models have less unused capacity

Table 6: Decision analysis of models in instances with $L = 10$ and $N = 20$.

	Γ	FVB	RFVB1	RFVB2	AARC	LAARC	ELAARC
# of instances with open facilities	1	75%	88%	93%	95%	95%	95%
	2	57%	86%	90%	94%	94%	94%
	5	10%	85%	85%	91%	91%	91%
Average # of open facility	1	1.56	1.66	1.86	1.86	1.86	1.86
	2	1.21	1.56	1.82	1.81	1.83	1.83
	5	1.20	1.49	1.61	1.62	1.70	1.71
Average total capacity	1	170227	167479	171089	164867	164797	164806
	2	146404	135752	156456	153134	153582	153905
	5	112252	78879	109826	119999	124363	125493

Table 7: Proportion of demand that is covered and total capacity that is unused averaged over 100 scenarios in a set of 100 problem instances.

Γ	Title	FVB	RFVB1	RFVB2	AARC	LAARC	ELAARC
1	Unused capacity (%)	1.18	2.06	1.52	0.74	0.74	0.74
	Covered demand (%)	63.11	72.46	79.25	79.05	79.02	79.03
2	Unused capacity (%)	0.17	0.79	0.74	0.20	0.21	0.22
	Covered demand (%)	41.93	58.15	70.85	73.02	73.21	73.36
5	Unused capacity (%)	0.00	0.00	0.02	0.00	0.00	0.00
	Covered demand (%)	5.80	34.23	47.43	55.55	57.60	58.12

and cover a larger percentage of demand than other models. Among the customer-driven model, the RFVB2 model appears to use a strategy that resembles much more to the strategies of the market-driven models.

6 Conclusion

In this paper, we have studied a two-stage robust location-transportation problem with demand uncertainty which was characterized using the budgeted uncertainty set. In order to overcome the known computational difficulty of resolution of this model, we presented five new conservative approximation models that each implement to a different extent the flexibility in the delayed production and transportation decisions. We believe these models, and in particular the RFVB2 and ELAARC models, are especially relevant to the transportation literature as the only conservative approximation model that had been presented prior to this work was the FVB model which as Example 1 and our empirical results demonstrated is severely flawed. While this flaw can easily be corrected for by adding a small amount flexibility to the delayed decisions as is done in the customer-driven RFVB2 model, the solution quality is drastically improved using market-driven models such as the ELAARC. On the other hand, improving solution quality through added flexibility necessarily comes at a price in terms of computational requirements for resolution. Table 8 summarizes this important trade-off. It is to be noted that we have good hopes that further research on more specialized algorithms for the LAARC and ELAARC models, possibly inspired from column generation method, might successfully improve the resolution time of these models.

A side product of our analysis is to have identified conditions under which full flexibility is not necessary in order to obtain a solution of the best quality possible. This is summarized in Table 9.

Finally, our numerical study compares the performances of the proposed approximation models against the performance of the FVB model and deterministic model in terms of sub-optimality of the approximate robust solution, resolution time, achievable risk-return trade-off, structure of optimal robust decisions, average customer coverage and average unused capacity.

Although our work focuses on a location-transportation problem, we expect our methods to be applicable to many other two-stage robust optimization problem with right-hand side uncertainty that appear in the

Table 8: Summary of the trade-off between flexibility of the decision rules, complexity of model, and quality of the solution.

Model	Variables				Observed
	P_i	Y_{ij}	θ_j	Total number	sub-optimality
ELAARC	$\sum_i Y_{ij}$	$\sum_k (X^+_{ijk} D^+_k + X^-_{ij} D^-_k) + W_{ij}$	$S^+_j D^+_j + S^-_j D^-_j$	$O(LN^2)$	6.3%
LAARC	$\sum_i Y_{ij}$	$\sum_k (X^+_{ijk} D^+_k + X^-_{ij} D^-_k) + W_{ij}$	0	$O(LN^2)$	12.7%
AARC	$\sum_i Y_{ij}$	$\sum_k X_{ijk} D_k + W_{ij}$	0	$O(LN^2)$	12.7%
RFVB2	$\sum_i Y_{ij}$	$X^+_{ij} D^+_j + X^-_{ij} D^-_j + W_{ij}$	0	$O(LN)$	100%
RFVB1	$\sum_i Y_{ij}$	$X_{ij} D_j + W_{ij}$	0	$O(LN)$	100%
FVB	P_i	$X_{ij} D_j$	0	$O(LN)$	100%

Table 9: Conditions for approximation models to identify optimal robust strategic decisions.

Condition	RC	FVB	RFVB1	RFVB2	AARC	LAARC	ELAARC
$C = 0$	×	×	✓	✓	✓	✓	✓
$\Gamma = 1$	×	×	×	×	×	✓	✓
$\Gamma = N$	✓	×	✓	✓	✓	✓	✓

field of transportation, such as network transportation problem, (e.g. Atamtürk and Zhang (2007)), supply chain network design problem, (e.g. Tsiakis et al. (2001)) and hub location-transportation problem (e.g. Oktal and Ozger (2013)).

Endnotes

1. While sources of uncertainty other than demand might affect the performance of facility location decisions and it might be interesting to account for them, in this paper we focus on demand uncertainty as we expect it to have the most impact on the quality of the decision that needs to be made. See for instance Delage et al. (in press) where the authors argue that simply using the expected values of parameters that appear in the objective function already generates solutions that can be considered robust.

2. Note that in this paper we will consider a model tractable if it can be reformulated as a mixed integer linear program of finite dimension.

3. To measure the optimal value of the TRLTP model, we employed the column and constraint generation algorithm proposed by Zeng and Zhao (2013).

APPENDIX

A Analytical solutions to RC and FVB models in Example 1

For the box uncertainty set, the RC model (2) takes the following form

$$\begin{aligned}
\underset{I,Z,Y,P}{\text{maximize}} \quad & \sum_i \sum_j (\eta - d_{ij}) Y_{ij} - cP - (C^T Z + K^T I) && (12a)\\
\text{subject to} \quad & \sum_i Y_{ij} \le \bar{D}_j - \hat{D}_j \,,\, \forall\, j, && (12b)\\
& \sum_j Y_{ij} \le P_i \,,\, \forall\, i, && (12c)\\
& P \le Z,\ Z \le MI && (12d)\\
& Y \in \mathbb{R}_+^{L\times N} \,, I_i \in \{0,1\}^L. && (12e)
\end{aligned}$$

In the optimal solution of RC model (12), the value of Y_{ij} is equal to zero, since $\eta - c_i - C_i - d_{ij} < 0$, for all i and j where $i \neq j$ and is equal to $\bar{D}_j - \hat{D}_j = 10000 - 5000 = 5000$ for all i and j when $i = j$. In sequence, the optimal value of variables P_i, Z_i and I_i are equal to 5000, 5000 and 1 for all i respectively. Therefore the optimal value of problem (12) is equal to $2,000$. On the other hand, the FVB model (5) with box uncertainty set takes the form as

$$\begin{aligned}
\underset{I,Z,X,P}{\text{maximize}} \quad & \sum_i \sum_j (\eta - d_{ij})(\bar{D}_j - \hat{D}_j) X_{ij} - \sum_i c_i P_i - \sum_i (C_i Z_i + K_i I_i) && (13a)\\
\text{subject to} \quad & \sum_i X_{ij} \le 1 \,,\, \forall\, j, && (13b)\\
& \sum_j (\bar{D}_j + \hat{D}_j) X_{ij} \le P_i \,,\, \forall\, i, && (13c)\\
& P_i \le Z_i \,,\, \forall\, i, && (13d)\\
& X_{ij} \ge 0 \,,\, \forall\, i, \forall\, j, && (13e)\\
& Z_i \le M I_i,\ I_i \in \{0,1\}\ \ \forall i. && (13f)
\end{aligned}$$

Similar to what we conclude above, the optimal value $X_{ij} = 0$ for all i and j when $i \neq j$. The optimal value of variable P_i is equal to $(\bar{D}_i + \hat{D}_i) X_{ii}$ for all i and the optimal value of variable Z_i is equal to that of variable P_i for all i. Therefore, the objective function (13a) can be reformulated as

$$\sum_i \eta(\bar{D}_i - \hat{D}_i) X_{ii} - \sum_i (c_i + C_i)(\bar{D}_i + \hat{D}_i) X_{ii} - \sum_i K_i 1_{\{X_{ii} > 0\}} = \sum_i (2000 X_{ii} - 3000 \times 1_{\{X_{ii}>0\}}) \le 0.$$

where the last inequality comes from $\sum_i X_{ij} \le 1$. Therefore the the optimal value of problem (13) is equal to zero in this example.

B Selecting large enough B for problem (10)

Lemma 1 *The problem* (10) *is equivalent to the problem* (4) *if*

$$B_j := \max_i (\eta - c_i - d_{ij})\ \ \forall j$$

Proof. Problem (4) can be reformulated using duality theory as

$$\begin{aligned}
h(I,Z) = \underset{\lambda,\theta}{\text{minimize}} \quad & Z^T \theta + D^T \lambda && (14a)\\
\text{subject to} \quad & \theta_i + \lambda_j \ge \eta - d_{ij} - c_i \,,\ \forall\, i, \forall\, j, && (14b)\\
& \lambda \in \mathbb{R}_+^N, \theta \in \mathbb{R}_+^L. && (14c)
\end{aligned}$$

It is implied from problem (14) that there is an optimal solution for which λ_j, dual variable associated to constraint (4b), is smaller or equal to $\max_i(\eta - c_i - d_{ij})$ for all j, therefore one can add to problem (14) the constraint $\lambda_j \leq \max_i(\eta - c_i - d_{ij})$ without affecting the optimal value. By applying duality theory a second time, we retrieve problem (10) with $\max_i(\eta - c_i - d_{ij})$ in place of every B_j as equivalent to problem (4). □

C Proof of Theorem 1

C.1 Proof of case $C = 0$

First, if all $C_i = 0$, then it is necessarily the case that all Z_i can be as large as MI_i. Next, we replace variables Y_{ij} by $X_{ij}D_j$ and P_i by $\sum_j D_j X_{ij}$ in the recourse problem (4) which makes the recourse problem equivalent to

$$\begin{aligned}
\underset{X}{\text{maximize}} \quad & \sum_i \sum_j (\eta - d_{ij} - c_i) X_{ij} D_j && \text{(15a)}\\
\text{subject to} \quad & \sum_i X_{ij} \leq 1 \,, \forall j, && \text{(15b)}\\
& \sum_j X_{ij} D_j \leq MI_i \,, \forall i, && \text{(15c)}\\
& X_{ij} \in \mathbb{R}_+^{L\times N}. && \text{(15d)}
\end{aligned}$$

where X_{ij} is the new second-stage decision variable, and . Constraint (15c) can be replaced by

$$X_{ij} \leq I_i \,, \forall i, \forall j, \qquad (16)$$

since (15c) implies that binary variable I_i is equal to 1 if there is a positive shipment from facility in location i to at-least one customer, therefore the objective function of the TRLTP can be reformulated as

$$\begin{aligned}
\min_{D\in\mathcal{D}} \quad \max_{X} \quad & \sum_i \sum_j (\eta - d_{ij} - c_i) X_{ij} D_j && \text{(17a)}\\
\text{subject to} \quad & \sum_i X_{ij} \leq 1 \,, \forall j, && \text{(17b)}\\
& X_{ij} \leq I_i \,, \forall i, \forall j, && \text{(17c)}\\
& X_{ij} \in \mathbb{R}_+^{L\times N}. && \text{(17d)}
\end{aligned}$$

Since both feasible sets for D and X are compact, based on Sion's theorem, we can reverse the order of minimization over $\mathcal{D}$ and maximization over X, therefore problem (3) with $C = 0$ can be reformulated as

$$\begin{aligned}
\underset{I,X}{\text{maximize}} \quad & -KI + \min_{D\in\mathcal{D}} \sum_i \sum_j (\eta - d_{ij} - c_i) X_{ij} D_j && \text{(18a)}\\
\text{subject to} \quad & \sum_i X_{ij} \leq 1 \,, \forall j, && \text{(18b)}\\
& X_{ij} \leq I_i \,, \forall i, \forall j, && \text{(18c)}\\
& X_{ij} \in \mathbb{R}_+^{L\times N}, I \in \{0,1\}^L. && \text{(18d)}
\end{aligned}$$

that is equivalent to RFVB1 model with $W = 0$, therefore RFVB1 model and all tighter approximation models are optimal. □

C.2 Proof of case $\Gamma = N$

Theorem 2 *(Ben-Tal et al., 2004) The adjustable robust counterpart of two-stage robust optimization problem is equivalent to its RC approximation where the uncertainty affecting every one of the constraints is independent of the uncertainty affecting all other constraints (constraint-wise uncertainty).*

According to this theorem, RC model (2) is optimal for box uncertainty set and based on Corollary 1, RFVB1, RFVB2, AARC, LAARC, ELAARC are optimal in this case and equivalent to the following formulation:

$$\underset{I,Z,Y,P}{\text{maximize}} \quad \sum_i \sum_j (\eta - d_{ij} - c_i) Y_{ij} - (C^T Z + K^T I) \tag{19a}$$

$$\text{subject to} \quad \sum_i Y_{ij} \leq \bar{D}_j - \hat{D}_j, \; \forall j, \tag{19b}$$

$$\sum_j Y_{ij} \leq Z_i \;, \forall i, \tag{19c}$$

$$Y \in \mathbb{R}_+^{L \times N}, \; I \in \{0,1\}^L \tag{19d}$$

□

C.3 Proof of case $\Gamma = 1$

Since the recourse function $h(I, Z, D)$ is concave in D and the budgeted uncertainty set is polyhedral, a worst-case demand necessarily occurs at one of the extreme points of $\mathcal{D}$. The TRLTP model is therefore equivalent to the following scenario based model in terms of optimal value and optimal solution for the decision variables I and Z:

$$\underset{I,Z,Y}{\text{maximize}} \quad -\sum_i (C_i Z_i + K_i I_i) + \psi \tag{20a}$$

$$\text{subject to} \quad \psi \leq \sum_i \sum_j (\eta - d_{ij} - c_i) Y_{ijl} \;\; \forall l \in \Omega, \tag{20b}$$

$$\sum_j Y_{ijl} \leq Z_i \;, \forall i, \; \forall l \in \Omega, \tag{20c}$$

$$\sum_i Y_{ijl} \leq D_j^l \;, \forall j, \; \forall l \in \Omega, \tag{20d}$$

$$Y_{ijl} \geq 0 \;, \forall i, \forall j, \; \forall l \in \Omega, \tag{20e}$$

$$Z_i \leq M I_i, \; \forall i, \; I_i \in \{0,1\} \; \forall i. \tag{20f}$$

where $\{D^l\}_{l \in \Omega}$ is the set of extreme point of the $\mathcal{D}$ uncertainty set.

A similar argument can be made of the LAARC model (9) which is equivalent to the following problem

$$\underset{I,Z,X^+,X^-,W}{\text{maximize}} \quad \psi - \sum_i (C_i Z_i + K_i I_i) \tag{21a}$$

$$\text{subject to} \quad \psi \leq \sum_i \sum_j (\eta - d_{ij} - c_i)(\sum_k X_{ijk}^+ D_k^{+l} + X_{ijk}^- D_k^{-l} + W_{ij}) \; \forall l \in \Omega, \tag{21b}$$

$$\sum_j (\sum_k X_{ijk}^+ D_k^{+l} + X_{ijk}^- D_k^{-l} + W_{ij}) \leq Z_i \;, \forall i, \; \forall l \in \Omega, \tag{21c}$$

$$\sum_i (\sum_k X_{ijk}^+ D_k^{+l} + X_{ijk}^- D_k^{-l} + W_{ij}) \leq D_j^l \,, \forall j, \;\; \forall l \in \Omega, \tag{21d}$$

$$\sum_k X_{ijk}^+ D_k^{+l} + X_{ijk}^- D_k^{-l} + W_{ij} \geq 0 \;, \forall i, \forall j, \; \forall l \in \Omega \tag{21e}$$

$$Z_i \leq M I_i, \; I_i \in \{0,1\} \;\; \forall i. \tag{21f}$$

where $\{(D^{+l}, D^{-l})\}_{l \in \Omega}$ is set of extreme points of $\mathcal{D}_2$.

There are $2N + 1$ extreme points in $\mathcal{D}$ when $\Gamma = 1$: i.e. the nominal demand as the first extreme point, in other extreme points all customers demand get nominal value except one where the demand can be either equal to its largest amount or lowest amount. Let us identify each of these extreme point as follows:

$$D^l = \begin{cases} \bar{D} & l = 0 \\ \bar{D} + e_l \hat{D} & l = 1, \ldots, N \\ \bar{D} - e_{l-N} \hat{D} & l = N+1, \ldots, 2N \end{cases} ,$$

where e_l is a vector with size N with all elements equal to 0 except for the lth element which is equal to 1. Therefore, the problem (20) with this uncertainty set can be shown to reduce to

$$\begin{aligned}
\underset{I,Z,Y}{\text{maximize}} \quad & -\sum_i (C_i Z_i + K_i I_i) + \psi && \text{(22a)}\\
\text{subject to} \quad & \psi \leq \sum_i \sum_j (\eta - d_{ij} - c_i) Y_{ijl} \;\; \forall\, l = 0, \ldots, 2N, && \text{(22b)}\\
& \sum_j Y_{ijl} \leq Z_i \,,\, \forall\, i,\; \forall\, l = 0, \ldots, 2N, && \text{(22c)}\\
& \sum_i Y_{ijl} \leq \bar{D}_j \,,\, \forall\, j,\; \forall\, l = 0, && \text{(22d)}\\
& \sum_i Y_{ijl} \leq \bar{D}_j + 1_{\{j=l\}} \hat{D}_j \,,\, \forall\, j,\; \forall\, l = 1, \ldots, N, && \text{(22e)}\\
& \sum_i Y_{ijl} \leq \bar{D}_j - 1_{\{j=l-N\}} \hat{D}_j \,,\, \forall\, j,\; \forall\, l = N+1, \ldots, 2N, && \text{(22f)}\\
& Y_{ij} \geq 0 \,,\, \forall\, i, \forall\, j,\; \forall\, l = 0, \ldots, 2N, && \text{(22g)}\\
& Z_i \leq M I_i,\; \forall\, i,\; I_i \in \{0,1\} \;\forall\, i, && \text{(22h)}
\end{aligned}$$

where the constraint (22d)–(22f) are equivalent to the constraint (20d), and where $l_{\{j=l\}}$ evaluates to 1 if $j = l$ and zero otherwise.

Let us now define the extreme points of $\mathcal{D}_2$ as:

$$(D^{+l}, D^{-l}) = \begin{cases} (0,0) & \forall l = 0, \\ (e_l \hat{D}, 0) & \forall l = 1, \ldots, N, \\ (0, e_{l-N} \hat{D}) & \forall l = N+1, \ldots, 2N, \end{cases} .$$

In particular, we should have

$$(D_k^{+l}, D_k^{-l}) = \begin{cases} (0,0) & \forall k,\; \forall l = 0, \\ (1_{\{k=l\}} \hat{D}_k, 0) & \forall k,\; \forall l = 1, \ldots, N, \\ (0, 1_{\{k=l-N\}} \hat{D}_k) & \forall k,\; \forall l = N+1, \ldots, 2N, \end{cases} .$$

Therefore, we can conclude that the application of the affine decision rules over $\mathcal{D}_2$ will reduce to

$$\sum_k X_{ijk}^{+} D_k^{+l} + \sum_k X_{ijk}^{-} D_k^{-l} + W_{ij} = \begin{cases} W_{ij} & \forall i,\; \forall j,\; \forall l = 0 \\ X_{ijl}^{+} \hat{D}_l + W_{ij} & \forall i,\; \forall j,\; \forall l = 1, \ldots, N \\ X_{i,j,l-N}^{-} \hat{D}_{l-N} + W_{ij} & \forall i,\; \forall j,\; \forall l = N+1, \ldots, 2N \end{cases} .$$

Problem (21) therefore reduces to

$$\begin{aligned}
\underset{I,Z,X^+,X^-,W}{\text{maximize}} \quad & \psi - \sum_i (C_i Z_i + K_i I_i) && \text{(23a)}\\
\text{subject to} \quad & \psi \leq \sum_i \sum_j (\eta - d_{ij} - c_i) W_{ij} \;\forall\, l = 0, && \text{(23b)}\\
& \psi \leq \sum_i \sum_j (\eta - d_{ij} - c_i)(X_{ijl}^{+} \hat{D}_l + W_{ij}) \;\forall\, l = 1, \ldots, N, && \text{(23c)}\\
& \psi \leq \sum_i \sum_j (\eta - d_{ij} - c_i)(X_{i,j,l-N}^{-} \hat{D}_{(l-N)} + W_{ij}) \;\forall\, l = N+1, \ldots, 2N, && \text{(23d)}\\
& \sum_j W_{ij} \leq Z_i \,,\, \forall\, i,\; \forall\, l = 0, && \text{(23e)}\\
& \sum_j (X_{ijl}^{+} \hat{D}_l + W_{ij}) \leq Z_i \,,\, \forall\, i,\; \forall\, l = 1, \ldots, N, && \text{(23f)}
\end{aligned}$$

$$\sum_j (X^-_{i,j,l-N}\hat{D}_{l-N} + W_{ij}) \le Z_i \ , \forall i, \ \forall l = N+1,\ldots,2N, \tag{23g}$$

$$\sum_i W_{ij} \le \bar{D}_j \ , \forall j, \ \ \forall l = 0, \tag{23h}$$

$$\sum_i (X^+_{ijl}\hat{D}_l + W_{ij}) \le \bar{D}_j + 1_{\{j=l\}}\hat{D}_j \ , \forall j, \ \ \forall l = 1,\ldots,N, \tag{23i}$$

$$\sum_i (X^-_{i,j,l-N}\hat{D}_{l-N} + W_{ij}) \le \bar{D}_j - 1_{\{j=l-N\}}\hat{D}_j \ , \forall j, \ \ \forall l = N+1,\ldots,2N, \tag{23j}$$

$$W_{ij} \ge 0 \ , \forall i, \forall j, \ \forall l = 0, \tag{23k}$$

$$X^+_{ijl}\hat{D}_l + W_{ij} \ge 0 \ , \forall i, \forall j, \ \forall l = 1,\ldots,N, \tag{23l}$$

$$X^-_{i,j,l-N}\hat{D}_{l-N} + W_{ij} \ge 0 \ , \forall i, \forall j, \ \forall l = N+1,\ldots,2N, \tag{23m}$$

$$Z_i \le MI_i, \ I_i \in \{0,1\} \ \ \forall i. \tag{23n}$$

In problem (23), we next reformulate the decision variables W_{ij}, X^+_{ijl} and $X^-_{i,j,l-N}$ as follows

$$W_{ij} \to \dot{Y}_{ij0}, \ \ \forall i, \ \forall j$$
$$X^+_{ijl} \to \frac{\dot{Y}_{ijl} - \dot{Y}_{ij0}}{\hat{D}_l}, \ \ \forall i, \ \forall j, \ \forall l = 1,\ldots,N,$$
$$X^-_{i,j,l-N} \to \frac{\dot{Y}_{ijl} - \dot{Y}_{ij0}}{\hat{D}_{(l-N)}}, \ \ \forall i, \ \forall j, \ \forall l = N+1,\ldots,2N,$$

therefore, problem (23) can be reformulated as

$$\underset{I,Z,\dot{Y}}{\text{maximize}} \quad \psi - \sum_i (C_i Z_i + K_i I_i) \tag{24a}$$

$$\text{subject to} \quad \psi \le \sum_i \sum_j (\eta - d_{ij} - c_i)\dot{Y}_{ij0} \ \ \forall l = 0, \tag{24b}$$

$$\psi \le \sum_i \sum_j (\eta - d_{ij} - c_i)\dot{Y}_{ijl} \ \ \forall l = 1,\ldots,N, \tag{24c}$$

$$\psi \le \sum_i \sum_j (\eta - d_{ij} - c_i)\dot{Y}_{ijl} \ \ \forall l = N+1,\ldots,2N, \tag{24d}$$

$$\sum_j \dot{Y}_{ij0} \le Z_i \ , \forall i, \ \forall l = 0, \tag{24e}$$

$$\sum_j \dot{Y}_{ijl} \le Z_i \ , \forall i, \ \forall l = 1,\ldots,N, \tag{24f}$$

$$\sum_j \dot{Y}_{ijl} \le Z_i \ , \forall i, \ \forall l = N+1,\ldots,2N, \tag{24g}$$

$$\sum_i \dot{Y}_{ijl} \le \bar{D}_j \ , \forall j, \ \ \forall l = 0, \tag{24h}$$

$$\sum_i \dot{Y}_{ijl} \le \bar{D}_j + 1_{\{j=l\}}\hat{D}_j \ , \forall j, \ \ \forall l = 1,\ldots,N, \tag{24i}$$

$$\sum_i \dot{Y}_{ijl} \le \bar{D}_j - 1_{\{j=l-N\}}\hat{D}_j \ , \forall j, \ \ \forall l = N+1,\ldots,2N, \tag{24j}$$

$$\dot{Y}_{ij0} \ge 0 \ , \forall i, \forall j, \ \forall l = 0, \tag{24k}$$

$$\dot{Y}_{ijl} \ge 0 \ , \forall i, \forall j, \ \forall l = 1,\ldots,N, \tag{24l}$$

$$\dot{Y}_{ijl} \ge 0 \ , \forall i, \forall j, \ \forall l = N+1,\ldots,2N, \tag{24m}$$

$$Z_i \le MI_i, \ I_i \in \{0,1\} \ \ \forall i. \tag{24n}$$

A careful comparison of problems (22) and (24) can confirm that these are the same so that they will return the same optimal value and identify the same set of optimal solutions for Z and I.

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