Accounting for Risk Measure Ambiguity when Optimizing Financial Positions

Erick Delage, Jonathan Y. Li

Tuesday, 9th of July, 2013
ICSP 2013, Bergamo
INTRODUCTION

- Since last financial crisis, there are no more “best practice” measures of risk
  - Variance assumes symmetric distribution and considers all random variables as risky, even positive ones
  - VaR ignores what happens in the tail of distributions and does not encourage diversification of risks
Since last financial crisis, there are no more “best practice” measures of risk

- Variance assumes symmetric distribution and considers all random variables as risky, even positive ones
- VaR ignores what happens in the tail of distributions and does not encourage diversification of risks

While the axioms proposed by Artzner et al. (1999) provide good guidance, it is never easy to choose which specific measure to use
Since last financial crisis, there are no more “best practice” measures of risk
  ▶ Variance assumes symmetric distribution and considers all random variables as risky, even positive ones
  ▶ VaR ignores what happens in the tail of distributions and does not encourage diversification of risks

While the axioms proposed by Artzner et al. (1999) provide good guidance, it is never easy to choose which specific measure to use

We propose a framework for accounting precisely for what is known of the risk preferences of a decision maker when optimizing financial positions
TABLE OF CONTENTS

INTRODUCTION

BACKGROUND
   Minimizing Risk
   Axioms of Risk Measures
   Preference Elicitation

METHODOLOGY
   Robust Optimization
   Robust Convex Risk Measure

EXPERIMENTS
   Details of experiments
   Numerical Results
   Conclusion
Table of Contents

Introduction

Background
  Minimizing Risk
  Axioms of Risk Measures
  Preference Elicitation

Methodology

Experiments
RISK MINIMIZATION

\[ Z = \xi^T x : \text{the return of a financial portfolio composed by a wealth allocation vector } x \text{ and securities with risky returns } \xi. \]

The goal is to solve

\[
\begin{align*}
\text{minimize} & \quad \rho(Z(x, \xi)) \\
\text{s.t.} & \quad \mathbb{E}[Z(x, \xi)] \geq \bar{r},
\end{align*}
\]

where \( \rho(\cdot) \) is a risk measure and \( \rho(Z_1) \geq \rho(Z_2) \) means portfolio \( Z_1 \) is perceived at least as risky as \( Z_2 \).


**RISK MINIMIZATION**

\[ Z = \xi^T x : \text{the return of a financial portfolio composed by a wealth allocation vector } x \text{ and securities with risky returns } \xi. \]

The goal is to solve

\[
\begin{align*}
& \text{minimize} & & \rho(Z(x, \xi)) \\
& \text{subject to} & & E \left[ Z(x, \xi) \right] \geq \bar{r},
\end{align*}
\]

where \( \rho(\cdot) \) is a risk measure and \( \rho(Z_1) \geq \rho(Z_2) \) means portfolio \( Z_1 \) is perceived at least as risky as \( Z_2 \).

Popular instances:

- **Variance**: \( \rho(Z) = E \left[ (Z - E[Z])^2 \right] \) : Markowitz (1952)
- **CVaR**: \( \rho(Z) = -1 \cdot E \left[ Z \mid Z \leq Z_{10\%} \right] \) : Rockafellar and Uryasev (2000)
**RISK MINIMIZATION**

\[ Z = \xi^Tx : \text{the return of a financial portfolio composed by a wealth allocation vector } x \text{ and securities with risky returns } \xi. \]

The goal is to solve

\[
\begin{align*}
\text{minimize} & \quad \rho(Z(x, \xi)) \\
\text{s.t.} & \quad \mathbb{E}[Z(x, \xi)] \geq \bar{r},
\end{align*}
\]

where \(\rho(\cdot)\) is a risk measure and \(\rho(Z_1) \geq \rho(Z_2)\) means portfolio \(Z_1\) is perceived at least as risky as \(Z_2\).

Which one and why?

- **Variance:** \(\rho(Z) = \mathbb{E}[(Z - \mathbb{E}[Z])^2] : \text{Markowitz (1952)}\)
- **CVaR:** \(\rho(Z) = -1 \cdot \mathbb{E}[Z|Z \leq Z_{10\%}] : \text{Rockafellar and Uryasev (2000)}\)
Axioms of Risk Measures

Convex risk measures (based on diversification principle) satisfy:

1. **Monotonicity**: if $X_1 \geq X_2$ then $\rho(X_1) \leq \rho(X_2)$;

2. **Translation Invariance**: if $c \in \mathbb{R}$, then
   
   $\rho(X_1 + c) = \rho(X_1) - c$,

3. **Convexity**:
   
   $\rho(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda \rho(X_1) + (1 - \lambda)\rho(X_2)$ \( \forall \lambda \in [0, 1] \).
Axioms of Risk Measures

Coherent risk measures (e.g. CVaR) satisfy:

1. **Monotonicity**: if $X_1 \geq X_2$ then $\rho(X_1) \leq \rho(X_2)$;
2. **Translation Invariance**: if $c \in \mathbb{R}$, then 
   $$\rho(X_1 + c) = \rho(X_1) - c,$$
3. **Convexity**: 
   $$\rho(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda \rho(X_1) + (1 - \lambda)\rho(X_2) \quad \forall \lambda \in [0, 1].$$
4. **Positive homogeneity**: $\rho(\lambda X) = \lambda \rho(X), \; \lambda \geq 0$
AXIOMS OF RISK MEASURES

CVaR satisfies:


Axioms of Risk Measures

CVaR satisfies:

1. **Monotonicity**: if $X_1 \geq X_2$ then $\rho(X_1) \leq \rho(X_2)$;
2. **Translation Invariance**: if $c \in \mathbb{R}$, then 
   
   $$\rho(X_1 + c) = \rho(X_1) - c,$$

3. **Convexity**:
   
   $$\rho(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda \rho(X_1) + (1 - \lambda)\rho(X_2) \quad \forall \lambda \in [0, 1].$$
4. **Positive homogeneity**: $\rho(\lambda X) = \lambda \rho(X), \quad \lambda \geq 0$
5. **Law invariance**: if $Z_i \sim F_i, Z_j \sim F_j$ and $F_i = F_j$, then 
   
   $$\rho(Z_i) = \rho(Z_j)$$

i.e. Risk only depends on distribution
Axioms of Risk Measures

Decision maker’s risk measure satisfies:

1. **Monotonicity**: if $X_1 \geq X_2$ then $\rho(X_1) \leq \rho(X_2)$;

2. **Translation Invariance**: if $c \in \mathbb{R}$, then $\rho(X_1 + c) = \rho(X_1) - c$;

3. **Convexity**:
   $$\rho(\lambda X_1 + (1 - \lambda) X_2) \leq \lambda \rho(X_1) + (1 - \lambda) \rho(X_2) \quad \forall \lambda \in [0, 1].$$

4. **Positive homogeneity**: $\rho(\lambda X) = \lambda \rho(X), \ \lambda \geq 0$

5. **Law invariance**: if $Z_i \sim F_i, Z_j \sim F_j$ and $F_i = F_j$, then $\rho(Z_i) = \rho(Z_j)$
   i.e. Risk only depends on distribution
**Axioms of Risk Measures**

Decision maker’s risk measure satisfies:

1. **Monotonicity**: if $X_1 \geq X_2$ then $\rho(X_1) \leq \rho(X_2)$;
2. **Translation Invariance**: if $c \in \mathbb{R}$, then
   
   $\rho(X_1 + c) = \rho(X_1) - c$,

3. **Convexity**:
   
   $\rho(\lambda X_1 + (1 - \lambda) X_2) \leq \lambda \rho(X_1) + (1 - \lambda) \rho(X_2) \quad \forall \lambda \in [0, 1]$.

4. **Positive homogeneity**: $\rho(\lambda X) = \lambda \rho(X)$, $\lambda \geq 0$

5. **Law invariance**: if $Z_i \sim F_i$, $Z_j \sim F_j$ and $F_i = F_j$, then
   
   $\rho(Z_i) = \rho(Z_j)$
   
   i.e. Risk only depends on distribution

6. **What else??**
**Axioms of Risk Measures**

Decision maker’s risk measure satisfies:

1. **Monotonicity**: if \( X_1 \geq X_2 \) then \( \rho(X_1) \leq \rho(X_2) \);

2. **Translation Invariance**: if \( c \in \mathbb{R} \), then
   \[
   \rho(X_1 + c) = \rho(X_1) - c,
   \]

3. **Convexity**:
   \[
   \rho(\lambda X_1 + (1 - \lambda) X_2) \leq \lambda \rho(X_1) + (1 - \lambda) \rho(X_2) \quad \forall \lambda \in [0, 1].
   \]

4. **Positive homogeneity**: \( \rho(\lambda X) = \lambda \rho(X) \), \( \lambda \geq 0 \)

5. **Law invariance**: if \( Z_i \sim F_i, Z_j \sim F_j \) and \( F_i = F_j \), then
   \[
   \rho(Z_i) = \rho(Z_j)
   \]
   i.e. Risk only depends on distribution

6. **What else?**
   
   ... We can ask or observe the DM.
The index of outcome $w_i$
1. Q1: $X_1 \succeq X_3$
Figure: Three risk profiles

1. **Q1**: $X_1 \succeq X_3$  
A1: Monotonicity $\Rightarrow \rho(X_1) \geq \rho(X_3)$
**Preference Elicitation**

![Graph showing three risk profiles](image)

**Figure: Three risk profiles**

1. **Q1**: $X_1 \succeq X_3$  
   **A1**: Monotonicity $\Rightarrow \rho(X_1) \geq \rho(X_3)$

2. **Q2**: $X_1 \succeq X_2$
The index of outcome \( w \)

**Figure: Three risk profiles**

1. **Q1**: \( X_1 \geq X_3 \) \hspace{1cm} **A1**: Monotonicity \( \Rightarrow \rho(X_1) \geq \rho(X_3) \)
2. **Q2**: \( X_1 \geq X_2 \) \hspace{1cm} **A2**: ??(we can learn from the DM)
What we know about $\rho$

1. Monotonicity
2. Translation Invariance
3. Convexity
4. Positive homogeneity
5. Law invariance
6. Elicitation Results: $\{\rho(X_j) \leq \rho(X_k)\}_{(j,k) \in I}$

Back to the problem:

$$\min_{x \in X} \rho(Z(x, \xi))$$

Question: How should we choose the portfolio $x$ when only the above information about $\rho$ is known?
TABLE OF CONTENTS

INTRODUCTION

BACKGROUND

METHODODOLOGY
   Robust Optimization
   Robust Convex Risk Measure

EXPERIMENTS
Robust Optimization Formulation

We propose the following minmax formulation

$$\min_{x \in \mathcal{X}} \sup_{\rho \in \mathcal{R}} \rho(Z(x, \xi)),$$

where $\mathcal{R} := \{\rho \mid \rho \text{ satisfies a subset of (1) to (6)}\}$

1. Monotonicity
2. Translation Invariance
3. Convexity
4. Positive homogeneity
5. Law invariance
6. Elicitation Results : $\{\rho(X_j) \leq \rho(X_k)\}_{(j,k) \in \mathcal{I}}$
THE ROBUST MEASURE AND OPTIMIZATION
Fact 1:
If $\rho$ is a convex/coherent/law inv. risk measure, then $\rho' = \sup_{\rho \in \mathcal{R}} \rho$ is also a convex/coherent/law inv. risk measure.
**The Robust Measure and Optimization**

**Fact 1:**
If \( \rho \) is a convex/coherent/law inv. risk measure, then 
\[ \rho' = \sup_{\rho \in \mathcal{R}} \rho \]
is also a convex/coherent/law inv. risk measure.

**Fact 2:**
Assuming that
- Set \( \mathcal{X} \) is convex
- Random vector \( \xi \) has \( N \) possible outcomes
then the risk vs. return optimization problem is a convex optimization problem that can be solved in polynomial time.
Robust Convex Risk Measure

Let’s consider the problem

\[
\begin{align*}
\text{minimize} & \quad \sup_{x \in \mathcal{X}} \sup_{\rho \in \mathcal{R}_1} \rho(Z(x, \xi)) , \\
\text{where } \mathcal{R}_1 := \{\rho \mid \rho \text{ satisfies conditions (1), (2), (3), and (6)}\}
\end{align*}
\]

1. Monotonicity
2. Translation Invariance
3. Convexity
4. Positive homogenity
5. Law invariance
6. Elicitation Results: \(\{\rho(X_j) \leq \rho(X_k)\}_{(j,k) \in \mathcal{I}}\)
**ROBUST CONVEX RISK MEASURE**

The optimization problem can be equivalently formulated as the following **finite dimensional convex optimization problem**

\[
(P) \quad \min_{x \in X, t, \theta} \quad t \\
\text{s.t.} \quad Z(x, \xi_i) + t \geq [X_1(\xi_i) \cdots X_m(\xi_i)]\theta + \delta^*^\top \theta, \quad \forall \ i = 1, \ldots, N \\
1^\top \theta = 1, \quad \theta \geq 0,
\]

where \(\delta^*\) is the optimal solution of

\[
\max_{\delta, \lambda} \quad \sum_i \delta_i \\
\text{s.t.} \quad \delta_j \leq \delta_k, \quad \forall (j, k) \in I \\
\delta_j \geq \delta_i - \lambda_i^\top (X_j - X_i), \quad \forall i, j \\
1^\top \lambda = 1 \quad \& \quad \lambda_i \geq 0, \quad \forall i
\]
We reformulate the problem from the perspective of acceptance sets (Föllmer and Schied 2002)

\[ \rho_A(Z) := \inf_{t \in \mathbb{R}} \{ t \mid Z + t \in A \}. \]
OUTLINE OF THE REFORMULATION PROCEDURE

We reformulate the problem from the perspective of acceptance sets (Föllmer and Schied 2002)

\[ \rho_A(Z) := \inf_{t \in \mathbb{R}} \{ t | Z + t \in A \}. \]

Our goal is to characterize the worst-case set \( A^* \) for a risk profile \( Z \)

\[ \sup_{A \in \mathbb{A}} \rho_A(Z), \]

where \( \mathbb{A} := \left\{ A \mid \rho_A = \text{convex risk measure} \right. \]

\[ \rho_A(X_j) \leq \rho_A(X_k), \forall (j, k) \in \mathcal{I} \} \] denotes a set of acceptance sets that are consistent with given information.
OUTLINE OF THE REFORMULATION PROCEDURE

We prove that the worst-case measure satisfies for all $Z$:

$$\sup_{A \in \mathcal{A}} \rho_A(Z) = \sup_{\delta \in \Delta} \left( \sup_{A \in \mathcal{A}(\delta)} \rho_A(Z) \right)$$

where

$$\mathcal{A}(\delta) := \left\{ A \mid \rho_A = \text{convex risk measure} \right\}$$

and

$$\Delta = \{ \delta \in \mathbb{R}^m \mid \mathcal{A}(\delta) \neq \emptyset \text{ and } \delta_j \leq \delta_k, \forall (j, k) \in \mathcal{I} \}$$
We prove that the worst-case measure satisfies for all $Z$:

$$
\sup_{A \in \Delta} \rho_A(Z) = \sup_{\delta \in \Delta} \sup_{A \in A(\delta)} \rho_A(Z) = \sup_{\delta \in \Delta} \rho_{\mathcal{H}(\delta)}(Z)
$$

where

$$\Delta = \{ \delta \in \mathbb{R}^m | A(\delta) \neq \emptyset \ & \delta_j \leq \delta_k, \ \forall (j, k) \in I \}$$

and $\mathcal{H}(\delta)$ is a convex polyhedron with the points $\{X_i + \delta_i\}$ as vertices

$$\mathcal{H}(\delta) = \left\{ y \in \mathbb{R}^N \mid \exists \theta \in \mathbb{R}^m, \ y \geq [X_1 \ X_2 \ \cdots \ X_m] \theta + \delta^T \theta, \ 1^T \theta = 1, \ \theta \geq 0 \right\}$$
We prove that the worst-case measure satisfies for all $Z$:

$$\sup_{A \in \mathcal{A}} \rho_A(Z) = \sup_{\delta \in \Delta} \rho_{\mathcal{H}(\delta)}(Z) = \rho_{\mathcal{H}(\delta^*)}(Z)$$

where $\mathcal{H}(\delta)$ is a convex polyhedron with the points $\{X_i + \delta_i\}$ as vertices

$$\mathcal{H}(\delta) = \left\{ y \in \mathbb{R}^N \left| \exists \theta \in \mathbb{R}^m, \begin{array}{c} y \geq [X_1 \ X_2 \ \cdots \ X_m] \theta + \delta^T \theta \\ \mathbf{1}^T \theta = 1, \ \theta \geq 0 \end{array} \right. \right\}$$
# Table of Contents

- **Introduction**
- **Background**
- **Methodology**
- **Experiments**
  - Details of experiments
  - Numerical Results
  - Conclusion
DETAILED OF EXPERIMENTS

We consider a static portfolio optimization problem with 4 assets over a period of one week

\[
\min_{x \geq 0, 1^T x = 1} \rho(R^T x)
\]

We simulate a decision maker’s true risk attitude using the following unknown law inv. coherent risk measure

\[
\rho := 0.1 \cdot \text{CVaR}_{20\%} + 0.9 \cdot \text{CVaR}_{95\%}.
\]
DETAILS OF EXPERIMENTS

We consider a static portfolio optimization problem with 4 assets over a period of one week

\[
\min_{x \geq 0, 1^T x = 1} \rho(R^T x)
\]

We simulate a decision maker’s true risk attitude using the following unknown law inv. coherent risk measure

\[
\rho := 0.1 \cdot \text{CVaR}_{20\%} + 0.9 \cdot \text{CVaR}_{95\%}.
\]

We compare

- Optimize knowing the true risk measure
DETAILS OF EXPERIMENTS

We consider a static portfolio optimization problem with 4 assets over a period of one week

$$\min_{x \geq 0, 1^T x = 1} \rho(R^T x)$$

We simulate a decision maker’s true risk attitude using the following unknown law inv. coherent risk measure

$$\rho := 0.1 \cdot \text{CVaR}_{20\%} + 0.9 \cdot \text{CVaR}_{95\%}.$$ 

We compare

- Optimize knowing the true risk measure
- Optimize assuming CVaR-20% or CVaR-95%
DETAILS OF EXPERIMENTS
We consider a static portfolio optimization problem with 4 assets over a period of one week

\[
\min_{x \geq 0, \mathbf{1}^\top x = 1} \rho(R^\top x)
\]

We simulate a decision maker’s true risk attitude using the following unknown law inv. coherent risk measure

\[
\rho := 0.1 \cdot \text{CVaR}_{20\%} + 0.9 \cdot \text{CVaR}_{95\%}.
\]

We compare

- Optimize knowing the true risk measure
- Optimize assuming CVaR-20\% or CVaR-95%
- Optimize assuming Convex Risk Measure
We consider a static portfolio optimization problem with 4 assets over a period of one week

$$\min_{x \geq 0, 1^\top x = 1} \rho(R^\top x)$$

We simulate a decision maker’s true risk attitude using the following unknown law inv. coherent risk measure

$$\rho := 0.1 \cdot \text{CVaR}_{20\%} + 0.9 \cdot \text{CVaR}_{95\%}.$$ 

We compare

- Optimize knowing the true risk measure
- Optimize assuming CVaR-20% or CVaR-95%
- Optimize assuming Convex Risk Measure
- Optimize assuming Coherent Risk Measure
We consider a static portfolio optimization problem with 4 assets over a period of one week

\[
\min_{x \geq 0, 1^\top x = 1} \rho(R^\top x)
\]

We simulate a decision maker’s true risk attitude using the following unknown law inv. coherent risk measure

\[
\rho := 0.1 \cdot \text{CVaR}_{20\%} + 0.9 \cdot \text{CVaR}_{95\%}.
\]

We compare

- Optimize knowing the true risk measure
- Optimize assuming CVaR-20\% or CVaR-95\%
- Optimize assuming Convex Risk Measure
- Optimize assuming Coherent Risk Measure
- Optimize assuming Law Invariant Convex Risk Measure
DETAILS OF EXPERIMENTS

We consider a static portfolio optimization problem with 4 assets over a period of one week

\[
\min_{x \geq 0, 1^T x = 1} \rho(R^T x)
\]

We simulate a decision maker’s true risk attitude using the following unknown law inv. coherent risk measure

\[
\rho := 0.1 \cdot \text{CVaR}_{20\%} + 0.9 \cdot \text{CVaR}_{95\%}.
\]

We compare

- Optimize knowing the true risk measure
- Optimize assuming CVaR-20\% or CVaR-95\%
- Optimize assuming Convex Risk Measure
- Optimize assuming Coherent Risk Measure
- Optimize assuming Law Invariant Convex Risk Measure
- Optimize assuming Law Invariant Coherent Risk Measure
DETAILS OF EXPERIMENTS

Use historical data about weekly returns of 14 assets from July 2007 to June 2012.

- On any given week, last 13 weeks’ returns constitute the outcome space
Use historical data about weekly returns of 14 assets from July 2007 to June 2012.

- On any given week, last 13 weeks’ returns constitute the outcome space
- For elicitation, we use a number of 13 weeks risk profiles from 2007 and 2008
DETAILS OF EXPERIMENTS

Use historical data about weekly returns of 14 assets from July 2007 to June 2012.

- On any given week, last 13 weeks’ returns constitute the outcome space
- For elicitation, we use a number of 13 weeks risk profiles from 2007 and 2008
- We report on 4000 experiments. For each one:
  - We randomly draw a date between 2009 and 2012
  - We randomly draw 4 assets for portfolio optimization
  - Performance of obtained portfolios is measured using true risk measure
Figure: Average perceived risk for the optimized portfolios over 4000 experiments
CONCLUSION

- Assuming a particular form of CVaR can be misleading, one can instead use an ambiguity averse risk measure formulation
- Impact of information about global attitude is significant but can be replaced with information about risk profiles
- The measures that account for law invariance can achieve nearly optimal performance with a small amount of additional information
CONCLUSION

- Assuming a particular form of CVaR can be misleading, one can instead use an ambiguity averse risk measure formulation
- Impact of information about global attitude is significant but can be replaced with information about risk profiles
- The measures that account for law invariance can achieve nearly optimal performance with a small amount of additional information

- In Armbruster and Delage (2012), we develop a similar framework but for expected utility theory