Robust Multi-Stage Decision Making

Erick Delage[1] **Dan Iancu**[2]

[1]CRC in Decision Making under Uncertainty
HEC Montréal

[2]Graduate School of Business
Stanford University

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Decisions Under Uncertainty

Decision maker chooses x; u unknown

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & C(x, u) \\ \text{s.t.} \quad & F(x, u) \geqslant 0 \end{aligned}$$

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Stochastic model:

$$\begin{aligned} &\min_{x} \mathbb{E}_{u}\left[C(x, u)\right] \\ &\text{s.t.} \;\; \mathbb{P}[\, F(x, u) \geqslant 0\,] \geqslant 1 - \epsilon \end{aligned}$$

- Distribution for u known
- Well-defined objective : *average* performance
- Good data, future like past, ...

Robust model:

$$\begin{aligned} &\min_{x} \max_{u \in \mathcal{U}} C(x, u) \\ &\text{s.t.} \;\; F(x, u) \geqslant 0, \forall\, u \in \mathcal{U} \end{aligned}$$

- Uncertainty set $\mathcal{U}$ known
- Well-defined objective : *worst-case* performance
- Poor data, non-stationarity, ...

Decisions Under Uncertainty

Decision maker chooses $\mathbf{x}$; $\mathbf{u}$ unknown

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \; & C(\mathbf{x}, \mathbf{u}) \\ \text{s.t.} \; & F(\mathbf{x}, \mathbf{u}) \geqslant 0 \end{aligned}$$

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- Distribution for $\mathbf{u}$ known
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- Uncertainty set $\mathcal{U}$ known
- Well-defined objective : *worst-case* performance
- Poor data, non-stationarity, ...
- *Infinitely many constraints*

Robust Optimization

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$$\min_{\boldsymbol{x}} \max_{\mathbf{u}\in\mathcal{U}} C(\boldsymbol{x},\mathbf{u})$$
$$\text{s.t. } F(\boldsymbol{x},\mathbf{u}) \geqslant 0, \forall\, \mathbf{u} \in \mathcal{U}$$

- For many classes of $\mathcal{U}$, can reformulate as convex optimization

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- For many classes of $\mathcal{U}$, can reformulate as convex optimization
  Soyster [1973], Falk [1976], Ben-Tal and Nemirovski [1998, 1999, 2000, 2002], Ben-Tal et al. [2002], El-Ghaoui and Lebret [1997], El-Ghaoui et al. [1998], Bertsimas and Sim [2003, 2004], Bertsimas et al. [2004], ...

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- Focus on *tractability*, *degree of conservatism*, *probabilistic guarantees*

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- Focus on *tractability*, *degree of conservatism*, *probabilistic guarantees*
- Some decisions *adjustable* → approach extended to dynamic settings
  Ben-Tal et al. [2004]

Robust Optimization

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Applications...

- inventory management e.g., [Ben-Tal et al., 2005, Bertsimas and Thiele, 2006, Bienstock and Özbay, 2008, ...]
- facility location and transportation [Baron et al., 2011, ...]
- scheduling [Lin et al., 2004, Yamashita et al., 2007, Mittal et al., 2014, ...]
- revenue management [Perakis and Roels, 2010, Adida and Perakis, 2006, ...]
- project management [Wiesemann et al., 2012, Ben-Tal et al., 2009, ...]
- energy generation and distribution [Zhao et al., 2013, Lorca and Sun, 2015, ...]
- portfolio optimization [Goldfarb and Iyengar, 2003, Tütüncü and Koenig, 2004, Ceria and Stubbs, 2006, Pinar and Tütüncü, 2005, Bertsimas and Pachamanova, 2008, ...]
- healthcare [Borfeld et al., 2008, Hanne et al., 2009, Chen et al., 2011, ...]

Book [Ben-Tal et al., 2009]; Review papers: [Bertsimas et al., 2011a, Gabrel et al., 2012]

Robust Optimization

Robust model:

$$\min_{\boldsymbol{x}} \max_{\mathbf{u}\in\mathcal{U}} C(\boldsymbol{x},\mathbf{u})$$
$$\text{s.t. } F(\boldsymbol{x},\mathbf{u}) \geqslant 0, \forall\, \mathbf{u} \in \mathcal{U}$$

Our objectives with this tutorial...

- Discuss static versus adjustable decisions
- Highlight some tractability issues
- Clarify the connection to robust dynamic programming
- Provide motivations for simple policies
- Illustrate time consistency issues

Outline

1. Introduction
2. When to Worry About Adjustable Decisions
3. The Adjustable Robust Counterpart Model
4. Connections With Robust Dynamic Programming (DP)
5. Simple Policies and Their Optimality
6. Should One Worry About Time Consistency?
7. Conclusions and Future Directions

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A Simple Inventory Problem

Consider the following inventory management problem:

$$\begin{aligned}
\underset{x,y}{\text{minimize}} \quad & \sum_{t=1}^{T} \left(\overbrace{c_t x_t}^{\text{ordering cost}} + \overbrace{h_t (y_{t+1})^+}^{\text{holding cost}} + \overbrace{b_t (-y_{t+1})^+}^{\text{backlog cost}} \right) \\
\text{s.t.} \quad & y_{t+1} = y_t + x_t - d_t, \ \forall\, t, && \text{(Stock balance)} \\
& 0 \leqslant x_t \leqslant M_t, \ \forall\, t, && \text{(Min/max order size)} \\
& y_1 = a \,, && \text{(Initial stock level)}
\end{aligned}$$

where

- x_t is number of goods ordered at time t and received at $t+1$
- y_t is number of goods in stock at beginning of time t
- d_t is demand between time t and $t+1$
- a is the initial inventory

A Linear Programming Formulation

This problem can be reformulated using the linear program

$$\begin{aligned}
\underset{x,y,s^+,s^-}{\text{minimize}} \quad & \sum_{t=1}^{T} \left(c_t x_t + h_t s_t^+ + b_t s_t^- \right) \\
\text{s.t.} \quad & s_t^+ \geqslant 0,\ s_t^- \geqslant 0\,,\ \forall\, t, \\
& s_t^+ \geqslant y_{t+1}\,,\ \forall\, t, \\
& s_t^- \geqslant -y_{t+1}\,,\ \forall\, t, \\
& y_{t+1} = y_t + x_t - d_t\,,\ \forall\, t, \\
& 0 \leqslant x_t \leqslant M_t\,,\ \forall\, t,
\end{aligned}$$

where

- s_t^+ is amount of goods held in storage during stage t
- s_t^- is amount of backlogged customer demands during stage t

A Linear Programming Formulation

This problem can be reformulated using the linear program

$$
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& y_{t+1} = y_t + x_t - d_t\,,\ \forall\, t, \\
& 0 \leqslant x_t \leqslant M_t\,,\ \forall\, t,
\end{aligned}
$$

How can we make this model robust to demand perturbations?

Naïve Robustification Scheme

Given that the vector of demand d is assumed to lie in some uncertainty set $\mathcal{U}$, let's consider the robust optimization model:

$$
\begin{aligned}
\underset{x,y,s^+,s^-}{\text{minimize}} \quad & \sum_{t=1}^{T} \left(c_t x_t + h_t s_t^+ + b_t s_t^- \right) \\
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\end{array}
$$

Unfortunately, this makes the model infeasible even when $|\mathcal{U}| = 2$:

$$
\left\{ \begin{array}{l} y_{t+1} = y_t + x_t - d_t^{(1)} \\ y_{t+1} = y_t + x_t - d_t^{(2)} \end{array} \right\} \Rightarrow\ d_t^{(1)} = d_t^{(2)}
$$

A Less Naïve Robustification Scheme

Robustify an alternate linear programming formulation:

$$\begin{aligned}
\underset{x,s^+,s^-}{\text{minimize}} \quad & \sum_{t=1}^{T} \left(c_t x_t + h_t s_t^+ + b_t s_t^- \right) \\
\text{s.t.} \quad & s_t^+ \geqslant 0,\ s_t^- \geqslant 0,\ \forall t, \\
& s_t^+ \geqslant y_1 + \sum_{t'=1}^{t} x_{t'} - d_{t'},\ \forall t, \\
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& 0 \leqslant x_t \leqslant M_t \ \forall t\ ,
\end{aligned}$$

where we simply replaced $y_{t+1} := y_1 + \sum_{t'=1}^{t} x_{t'} - d_{t'}$ in order to capture the fact that stock level evolves according to demand.

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& 0 \leqslant x_t \leqslant M_t, \forall\, t\ .
\end{aligned}
$$

Still two issues remain:

1. the orders should be adjustable w.r.t. the observed demand
2. (s_t^+, s_t^-) should be fully adjustable (more subtle)

Why (s_t^+, s_t^-) Should Be Fully Adjustable

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Consider the two-stage problem:

<u>Deterministic model</u>:

$$
\begin{aligned}
\min_{x_1} \quad & \overbrace{0.5x_1}^{\text{ordering}} + \overbrace{(x_1 - d_1)^+}^{\text{holding}} \\
& \quad + \underbrace{(d_1 - x_1)^+}_{\text{backlog}} \\
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<u>Less naïve robust model</u>:

$$\begin{aligned} \min_{x_1, s_1^+, s_1^-} \quad & 0.5x_1 + s_1^+ + s_1^- \\ \text{s.t.} \quad & s_1^+ \geqslant 0,\ s_1^- \geqslant 0 \\ & s_1^+ \geqslant x_1 - d_1 \,,\ \forall\, d_1 \in [0,2] \\ & s_1^- \geqslant d_1 - x_1 \,,\ \forall\, d_1 \in [0,2] \\ & 0 \leqslant x_1 \leqslant 2 \,. \end{aligned}$$

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Less naïve robust model:

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$$\text{s.t.} \quad 0 \leqslant x_1 \leqslant 2 \ .$$

Conclusions:

- Less naïve robust model states $x_1^* := 0$, $s_1^{+*} := 0$, and $s_1^{-*} := 2$ with worst-case cost of 2

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- Alternatively, $x_1^{**} := 1$ achieves a total cost lower than 1.5 for all $d_1 \in [0, 2]$

Why (s_t^+, s_t^-) Should Be Fully Adjustable

Consider the two-stage problem:

<u>Deterministic model</u>:

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An Accurate Two-stage Robust Inventory Model

The robust two-stage problem actually takes the form:

$$\begin{aligned} \underset{x_1}{\text{minimize}} \quad & \sup_{d_1 \in [0,2]} \; 0.5x_1 + h(x_1, d_1) \\ \text{s.t.} \quad & 0 \leqslant x_1 \leqslant 2 \,, \end{aligned}$$

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- Note how this form accounts for the fact that s_1^+ and s_1^- might jointly depend on the realized value of d_1.

Alternate Representation of Less Naïve Robust Model

Comparatively, the less naïve robust model was solving:

$$\begin{aligned} \underset{x_1, s_1^+, s_1^-}{\text{minimize}} \quad & \sup_{d_1 \in [0,2]} \; 0.5x_1 + g(x_1, s_1^+, s_1^-, d_1) \\ \text{s.t.} \quad & s_1^+ \geqslant 0, \; s_1^- \geqslant 0 \\ & 0 \leqslant x_1 \leqslant 2 \; , \end{aligned}$$

where

$$g(x_1, s_1^+, s_1^-, d_1) := \begin{cases} s_1^+ + s_1^- & \text{if } s_1^+ \geqslant x_1 - d_1 \text{ and } s_1^- \geqslant d_1 - x_1 \\ \infty & \text{otherwise} \end{cases} \quad .$$

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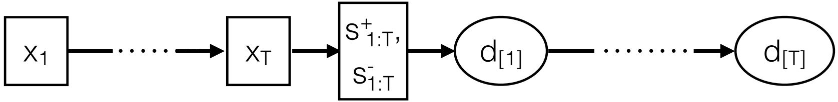
Timing characterized by less naïve robust model



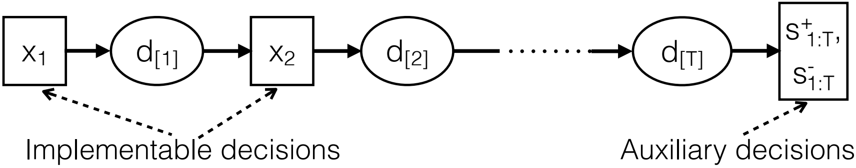
Takeaway Message about Adjustable Decisions

When robustifying decision models that either involve

- "implementable" decisions at different time periods (e.g. x_t)
- "auxiliary" decisions such as (s_t^+, s_t^-) that are used to assess overall performance of implemented decisions

one needs to carefully identify the chronology of decisions and observations and employ the adjustable robust counterpart framework introduced in (Ben-Tal et al., 2004)

Accurate timing of decisions and observations in inventory problem



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Applying the ARO Framework to Inventory Management

Applying the ARO Framework to Inventory Management

1. Formulate a deterministic model where implementable and auxiliary decisions can be identified:

$$
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\text{s.t.} \quad & s_t^+ \geqslant 0,\ s_t^- \geqslant 0,\ \forall t, \\
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\end{aligned}
$$

2. Identify an accurate chronology:

$$
x_1 \ \rightarrow\ d_{[1]}\ \rightarrow\ x_2\ \rightarrow \cdots \rightarrow\ x_T\ \rightarrow\ d_{[T]}\ \rightarrow\ (s_{1:T}^+, s_{1:T}^-)
$$

Applying the ARO Framework to Inventory Management

3. Describe the adjustable robust counterpart

$$\begin{aligned}
\min_{\substack{x_1, \{x_t(\cdot)\}_{t=2}^T, \\ \{s_t^+(\cdot), s_t^-(\cdot)\}_{t=1}^T}} \quad & \sup_{d \in \mathcal{U}} \sum_{t=1}^{T} \left(c_t x_t(d_{[t-1]}) + h_t s_t^+(d) + b_t s_t^-(d) \right) \\
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\end{aligned}$$

where

- Each $x_t : \mathbb{R}^{t-1} \to \mathbb{R}$ is adjusted to the observed demand $d_{[t-1]}$
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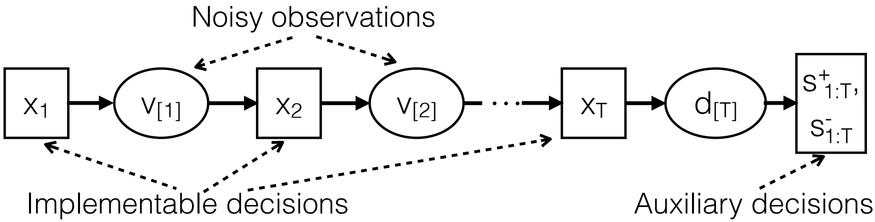
Robust Inventory Management with Noisy Observations

In some situations, the observations that are made are noisy.



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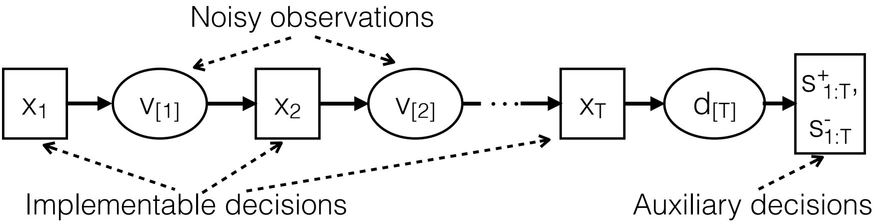

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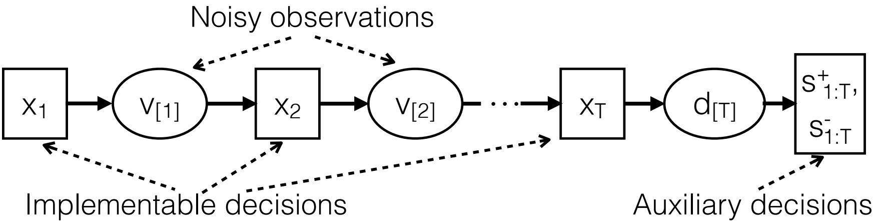

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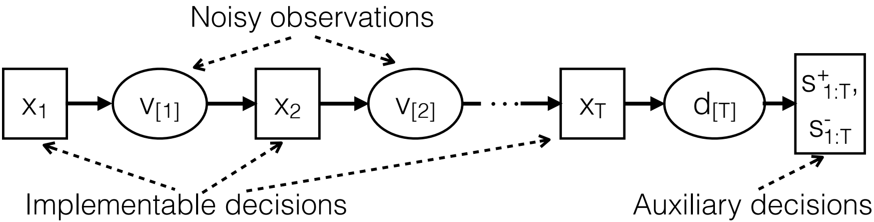

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  e.g. $v_t = d_t - \xi_t$ with $\xi_t \in [-d_t, d_t]$., with $\sum_t |\xi_t| \leqslant \Gamma$.

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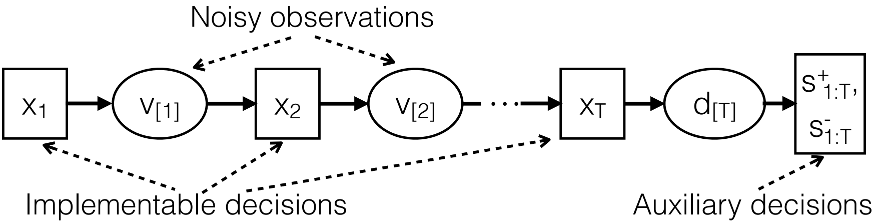

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See (de Ruiter et al., 2014) for detailed methodology.

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$$
\mathcal{U}' := \left\{ (d,v) \in \mathcal{U} \times \mathbb{R}^T \;\middle|\; \exists\, \xi \in \mathbb{R}_+^T,\; v_t = d_t - \xi_t,\; |\xi_t| \leqslant d_t,\; \sum_t |\xi_t| \leqslant \Gamma \right\}
$$

NP-Hardness of Adjustable Robust Optimization

Theorem (Ben-Tal et al., 2004):

Solving a robust multi-stage linear programming model is NP-hard.

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Sketch of proof:

- Consider the two-stage ARO model where $x : \mathbb{R}^m \to \mathbb{R}^N$:

$$\begin{aligned} \underset{x(\cdot)}{\text{minimize}} \quad & \sup_{u \in [0,1]^m} \sum_{i=1}^{N} (x_i(u) - 1) \\ \text{s.t.} \quad & x_i(u) \geqslant a_{i,k}^\top u + b_{i,k}\,,\ \forall\, u \in [0,1]^m\,, \begin{cases} \forall\, i = 1, \ldots, N\,, \\ \forall\, k = 1, 2, 3. \end{cases} \end{aligned}$$

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- Checking if optimal value is $\geqslant 0$ can be as difficult as a 3-SAT problem with m variables and N clauses like $v_{j_1} \vee v_{j_2} \vee \bar{v}_{j_3}$.
  - E.g., choose (a, b) such that:

$$\begin{aligned} x_i^*(z) &= \max_k a_{i,k}^\top u + b_{i,k} = \max\{u_{j_1}; u_{j_2}; 1 - u_{j_3}\} \\ &= 1 \text{ if } u_{j_1} = 1,\ u_{j_2} = 1, \text{ or } u_{j_3} = 0, \text{ otherwise it's } < 1. \end{aligned}$$

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- nested min-max problems
- proper updating rule: *projection* (analogous to "conditioning")

Bellman Principle; Robust DP Recursions

- The *state* of the system at time t:

$$S_t := \left[y_t \;\; d_{[t-1]}^\top\right]^\top = \left[y_t \;\; d_1 \;\; d_2 \;\; \dots \;\; d_{t-1}\right]^\top \in \mathbb{R}^t$$

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$$\mathcal{U}_{\text{h-cube}} = \times_{t=1}^{T}[\underline{d}_t, \bar{d}_t] \quad \rightarrow \quad S_t = y_t$$

$$\mathcal{U}_{\text{budget}} = \left\{d : \exists\, z, \|z\|_\infty \leqslant 1, \|z\|_1 \leqslant \Gamma, d_t = \bar{d}_t + \hat{d}_t z_t\right\} \rightarrow S_t = \left[y_t, \sum_{\tau=1}^{t-1} |z_\tau|\right]$$

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Observations:

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2. When $\mathcal{U}$ has special structure, can reduce state space
   - Reduce computational burden
   - Prove structural results, comparative statics

$$x_t^*(y) = \min\big(M_t, \max(0, \ \theta_t - y)\big) \qquad \text{(modified) } \textit{base-stock policy}$$

A More Subtle Point...

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Consider this problem:

$$J^* = \max_{d \in [\underline{d}, \bar{d}]} \min_{0 \leqslant x} f(d, x)$$

$$\big(y_1 \to d_1 \equiv d \to y_2 \equiv y(d) \to x_2 \equiv x\big)$$

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Is $x^*(d)$ necessary for achieving J^*?
Is Bellman optimality necessary in robust dynamic problems?

Is Bellman Optimality **Necessary?**

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Define the set of *worst-case optimal* policies:

$$\mathcal{X}^{\mathrm{wc}} := \left\{ x : [\underline{d}, \bar{d}] \to \mathbb{R}^+ \ : \ f\big(d, x(d)\big) \leqslant J^*, \ \forall\, d \in \mathcal{U} \right\}.$$

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- In fact, $\mathcal{X}^{\text{wc}}$ very degenerate: infinitely many policies

  (e.g., $\lambda\, x^* + (1 - \lambda) x^{\text{aff}}$, for any $\lambda \in [0, 1]$)

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- This degeneracy is typical for robust multi-stage problems
  ("If adversary does not play optimally, you don't have to, either...")

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- Worst-case optimal policies *must be implemented with resolving*

Table of Contents

Static Policies

- Simplest possible policies : static (i.e., non-adjustable)
- Very tractable
- Optimal for LPs with *row-wise* uncertainty [Ben-Tal et al., 2009]

$$\min_{x(z)} \quad \max_{z_0 \in \mathcal{Z}_0} \left[c(z_0)^\top x(z) + d(z_0) \right]$$
$$\text{subject to} \quad a_j(z_j)^\top x(z) \leqslant b_j(z_j), \ \forall z_j \in \mathcal{Z}_j, \ \forall j = 1, \ldots, J.$$

- Result extended recently [Bertsimas et al., 2015]
- Good performance under other uncertainty sets [Bertsimas and Goyal, 2010, Bertsimas et al., 2011b, ...]

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& 0 \leqslant x_t + X_t d \leqslant M_t\,, \ \forall\, d \in \mathcal{U},
\end{aligned}
$$

- Decision variables: coefficients $\mathcal{X} = \left\{x_t^0, X_t, s_t^+, S_t^+, s_t^-, S_t^-\right\}_{t=1}^{T}$
  - $X_t \in \mathbb{R}^{t-1} \times 0^{T-t+1}$ to ensure non-anticipativity

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& s_t^- + S_t^- d \geqslant -y_1 - \sum_{\tau=1}^{t} (x_\tau^0 + X_\tau d - d_\tau), \ \forall\, d \in \mathcal{U}, \\
& 0 \leqslant x_t + X_t d \leqslant M_t\,, \ \forall\, d \in \mathcal{U},
\end{aligned}$$

- **Two** layers of sub-optimality: **policy** *and* **auxiliary variables**

Affine Decision Rules

- Restrict attention to *affine* decision rules [Ben-Tal et al., 2004]
- *Affinely Adjustable Robust Counterpart* approximation scheme
- Tractable under fixed recourse [Ben-Tal et al., 2004]
- Excellent performance in a variety of applications [Ben-Tal et al., 2005, Mani et al., 2006, Adida and Perakis, 2006, Babonneau et al., 2010, ...]
- Optimal for linear problems with simplex uncertainty [Ben-Tal et al., 2009]
- Optimal for our multi-period inventory model under hypercube uncertainty set $\mathcal{U} = \times_{t=1}^{T} [\underline{d}_t, \bar{d}_t]$ [Iancu et al., 2013, Bertsimas et al., 2010]

Piecewise Affine Rules

- One can also use piece-wise affine rules, e.g.,

$$x_t(d_{[t-1]}) := x_t^0 + X_t d_{[t-1]} + \sum_{k=1}^{t-1} \theta_{tk}^{+}(d_k)^{+} + \theta_{tk}^{-}(-d_k)^{+}$$

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- These policies are actually affine in $\xi := [d, d^+, d^-]$
  - It would suffice to have a tractable representation for:

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- Such representations known for $\mathcal{U}_{\text{h-cube}}$, $\mathcal{U}_{\text{budget}}$, and other sets based on "absolutely symmetric convex functions" [Ben-Tal et al., 2009]

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- Large literature on other (nonlinear) rules:
  - *segregated rules* [Chen and Zhang, 2009, Goh and Sim, 2010, **?**]
  - *piecewise constant* [Bertsimas and Caramanis, 2010, Bertsimas et al., 2011b]
  - *kernelized* [Chatterjee et al., 2011, Skaf and Boyd, 2009]
  - *polynomial* [Ben-Tal et al., 2009, Bertsimas et al., 2011c]

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Definition of time consistency:

Axiomatic property that requires a decision maker's stated preferences over future courses of action to remain consistent with the actual preferred actions when planned-for contingencies arise.

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- Relates to the way multi-stage decision models are employed to generate implemented decisions as the future unfolds
- Good news: All robust multi-stage decision models naturally express stage-wise preferences that are time consistent
- Bad news: When decisions are implemented in a shrinking horizon fashion, the uncertainty set must be updated according to the "proper" update rule in order to give rise to this said consistency

Time Consistency in Robust “Coffee-vendor” Problem

Robust coffee-vendor problem (Part I):

A coffee stand is operated for one morning in the lobby of two hotels:

- hotel #1 during 7am-9am
- hotel #2 during 9am-11am
- possibility to replenish with fresh coffee at 9am

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This situation seems to motivate the following uncertainty set :

$$\mathcal{U} := \{d \in [0, 2]^2 \mid d_1 + d_2 \leqslant 3\}$$

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Robust coffee-vendor problem (Part II):

The coffee-vendor signed a contract with the two hotels to serve coffee to every thirsty guests. Here are some financial facts:

- contracts' revenue = 600\$ in total upfront
- production cost at 6am = 1\$ per cup
- production cost at 9am = 4\$ per cup
- salvage cost = 0\$ per cup
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$$\underset{x_1 \geqslant 0, x_2 \geqslant 0}{\text{minimize}} \quad \underbrace{x_1 + 4x_2 + 10(d_1 + d_2 - x_1 - x_2)^+}_{\text{total cost (in hundreds of \$)}}$$

Robust Two-stage Coffee-vendor Problem

$$\begin{aligned} \underset{x_1, x_2(\cdot)}{\text{minimize}} \quad & \sup_{d \in \mathcal{U}} \; x_1 + 4x_2(d_1) + 10(d_1 + d_2 - x_1 - x_2(d_1))^+ \\ \text{s.t.} \quad & x_1 \geqslant 0, \;\; x_2(d_1) \geqslant 0 \,, \; \forall \, d \in \mathcal{U} \,, \end{aligned}$$

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Conclusions that can be drawn based on optimal solution:

- Prepare 3 (hundred) cups at 6am
- Don't refresh the coffee at 9am under any circumstances
- The expected worst-case cost is 300\$ under any circumstances
- This gives an anticipated worst-case profit of $600\$ - 300\$ = 300\$$

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Is this truly the worst-case profit that will be achieved ?

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$\boxed{\text{No surprises.}}$

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> This issue can arise any time a new confidence region is used to update the uncertainty set dynamically.

One Possible Fix Using Bi-level Modeling

The following model accounts explicitly for time inconsistency

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This model suggests preparing 4 (hundred) cups at 6am, and nothing later. Under scenario #2, the worst-case profit is 200$!!!

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The question remains of how to efficiently resolve these types of models...

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Should one account for inconsistency directly in the decision model?

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- In any case, it is important to acknowledge these issues in reporting decisions that are prescribed by these models

Future Directions

- Full theoretical understanding of quality of simple policies
- Development of decomposition schemes for accelerating resolution of robust multi-stage problems
- Identify tractable conservative approximation methods for problems with integer or random recourse variables
- Advanced methods that provide the decision maker with better control of temporal correlation, level of conservatism, and time consistency

Thank you for attending! Questions?...

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