The Linear Programming Approach to Approximate Dynamic Programming. [2, 3]

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March 28, 2006

Outline

Motivation

- Curse of dimensionality makes exact solution of large MDPs intractable
- Interest in approximate DP has grown lately due to some success stories...
- but with significant trial and error and poor generalization
- An LP formulation may hopefully yield theoretical results

Problem formulation

- Finite state space: $\mathcal{S}$, $|\mathcal{S}| = N$
- $\forall x \in \mathcal{S}$ there exists a finite set of actions $\mathcal{A}_x$
- Taking action $a \in \mathcal{A}_x$ yields cost $g_a(x)$
- State transition probabilities: $p_a(x, y).\forall x \in \mathcal{S}.y \in \mathcal{S}$
- With policy u we have: $p_{u(x)}(x, y)$. Consider transition matrix P_u whose (x, y)th entry is $p_{u(x)}(x, y)$

Optimality criterion

- Optimize infinite-horizon discounted cost:

$$J_u(x) = E\left[\sum_{t=0}^{\infty} \alpha^t g_u(x_t) | x_0 = x\right]$$

- Well known there exists a single policy u that minimizes $J_u(x)$ simultaneously for all x
- The goal is to find that single policy

DP Operator T

- Define the DP operators T_u and T as:

$$\begin{aligned} T_u J &= g_u + \alpha P_u J \\ TJ &= \min_u (g_u + \alpha P_u J) \end{aligned} \qquad (1)$$

- The solution of Bellman's equation is $J = TJ$
- The unique solution J^* of (1) is the optimal cost-to-go function: $J^* = \min_u J_u$
- Optimal actions generated by:

$$u(x) = \underset{a \in \mathcal{A}_x}{\text{argmin}} \left(g_a(x) + \alpha \sum_{y \in \mathcal{S}} p_a(x,y) J^*(y) \right)$$

Linear Programming Approach (1/2)

- One approach to solve Bellman's equation:

$$\begin{aligned} \max \quad & c^T J, \\ \text{s.t.} \quad & TJ \geq J \end{aligned}$$

  c is a vector with positive *state-relevance weights*
- Can be shown that any feasible J satisfies $J \leq J^*$
- It follows that for any c, J^* is the unique solution to the above equation

Linear Programming Approach (2/2)

- T is a nonlinear operator
- We can rewrite problem as:

$$\begin{array}{ll} \max & c^T J \\ \text{s.t.} & g_a(x) + \alpha \sum_{y \in \mathcal{S}} p_a(x,y) J(y) \geq J(x) \\ & \forall x \in \mathcal{S}. \forall a \in \mathcal{A}_x \end{array}$$

- This problem will be referred to as the *exact LP*
- Any realistic problem will have a large number of variables and constraints!

LP Approach to approximate DP (1/2)

- Given pre-selected basis functions $\phi_1, \cdots, \phi_K$, define Φ as:

$$\Phi = \begin{bmatrix} | & & | \\ \phi_1 & \vdots & \phi_K \\ | & & | \end{bmatrix}$$

- Want to compute a weight vector $\tilde{r} \in \mathcal{R}^K$ s.t. $\Phi\tilde{r} \approx J^*$
- Policy defined according to

$$u(x) = \underset{a \in \mathcal{A}_x}{\text{argmin}} \left(g_a(x) + \alpha \sum_{y \in \mathcal{S}} p_a(x, y)(\Phi\tilde{r})(y) \right)$$

would hopefully be near-optimal

LP Approach to approximate DP (2/2)

- As before, can reformulate LP as:

$$\begin{aligned} \max \quad & c^T \Phi r \\ \text{s.t.} \quad & g_a(x) + \alpha \sum_{y \in \mathcal{S}} p_a(x,y)(\Phi r)(y) \geq (\Phi r)(x) \\ & \forall x \in \mathcal{S}. \forall a \in \mathcal{A}_x \end{aligned} \tag{2}$$

- This problem will be referred to as the *approximate LP*
- Number of variables reduced to K, but number of constraints remains as large as before

Importance of state-relevance weights

- In the exact LP, maximizing $c^T J$ yields J^* for any choice of c
- The same is not true for the approximate LP

Lemma

A vector $\tilde{r}$ solves

$$\begin{array}{rc} \max & c^T \Phi r \\ \textit{s.t.} & T\Phi r \geq \Phi r \end{array}$$

if and only if it solves

$$\begin{array}{rc} \min & \|J^* - \Phi r\|_{1,c} \\ \textit{s.t.} & T\Phi r \geq \Phi r \end{array}$$

- The algorithm can be lead to generate better approximations in a certain region of the state space by assigning a larger weight to that region!

Measuring quality of policies (1/2)

- If ν is the initial state distribution, a measure of the quality of policy u is: $E_{X\sim\nu}\left[J_u(X) - J^*(X)\right] = \|J_u - J^*\|_{1,\nu}$
- Define a measure $\mu_{u,\nu}$ over state space associated with policy u and distribution ν given by

$$\begin{aligned} \mu_{u,\nu}^T &= (1-\alpha)\nu^T \sum_{t=0}^{\infty} \alpha^t P_u^t \\ &= (1-\alpha)\nu^T (I - \alpha P_u)^{-1} \end{aligned}$$

- $\mu_{u,\nu}$ captures expected frequency of visits to each state when system runs under policy u, conditioned on initial state distributed according to ν
- It turns out that $\mu_{u,\nu}$ is a probability distribution

Measuring quality of policies (2/2)

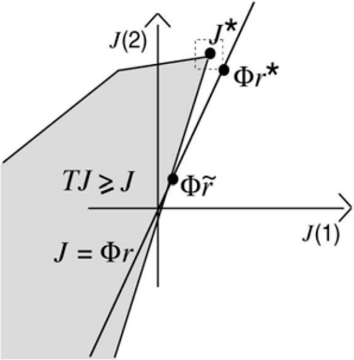
Theorem

Let $J : \mathcal{S} \mapsto \mathcal{R}$ be such that $TJ \geq J$. Then

$$\|J_{u_J} - J^*\|_{1,\nu} \leq \frac{1}{1-\alpha}\|J - J^*\|_{1,\mu_{u_J,\nu}}$$

- The above theorem says that if the approximate cost-to-go function J is close to J^*, the performance of the policy generated by J should also be close to the performance of the optimal policy
- We may want to choose c so that it captures frequency with which different states are visited (which in general depends on policy being used)

Error bounds for the approximate LP



- Would like to guarantee that $\Phi\tilde{r}$ is not too much farther from J^* than Φr^* is

A Simple Bound

Theorem

Let e be in the span of the columns of Φ and c be a probability distribution. Then, if $\tilde{r}$ is an optimal solution to the approximate LP,

$$\|J^* - \Phi\tilde{r}\|_{1,c} \leq \frac{2}{1-\alpha} \min_r \|J^* - \Phi r\|_\infty$$

- Establishes that when the optimal cost-to-go function lies close to the span of the basis functions, the approximate LP generates a good approximation.
- However, $\min_r \|J^* - \Phi r\|_\infty$ is typically huge in practice
- Also, the above bound doesn't take the choice of c into account

Lyapunov Functions

- Introduce operator H for all $V : \mathcal{S} \mapsto \mathcal{R}$ as:

$$(HV)(x) = \max_{a \in \mathcal{A}_x} \sum_y P_a(x,y)V(y)$$

- For each $V : \mathcal{S} \mapsto \mathcal{R}$, define a scalar β_V by

$$\beta_V = \max_x \frac{\alpha(HV)(x)}{V(x)}$$

- Denote $V : \mathcal{S} \mapsto \mathcal{R}^+$ a **Lyapunov function** if $\beta_V < 1$
- Equivalent to condition that there exist $V > 0$ and $\beta < 1$ s.t. $\alpha(HV)(x) \leq \beta V(x), \quad \forall x \in \mathcal{S}$
- β_V conveys a degree of ”stability”, with stronger values representing stronger stability

An Improved Bound

Theorem

Let $\tilde{r}$ be a solution of the approximate LP. Then, for any $v \in \mathcal{R}^K$ such that $(\Phi v)(x) > 0$ for all $x \in \mathcal{S}$ and $\alpha H \Phi v < \Phi v$,

$$\|J^* - \Phi \tilde{r}\|_{1,c} \leq \frac{2c^T \Phi v}{1 - \beta_{\Phi v}} \min_r \|J^* - \Phi r\|_{\infty, 1/\Phi v}$$

- With introduction of $\|\cdot\|_{\infty,1/\Phi v}$, the error at each state is now weighted by the reciprocal of the Lyapunov function value.
- The Lyapunov function should take on large values in undesirable regions of state space (where J^* is large
- State relevance weights are now factored into new bound

The Constraint Sampling Strategy - I

Consider the approximate LP:

$$\begin{aligned} \text{maximize} \quad & c^T \Phi r, \\ \text{subject to} \quad & T\Phi r \geq \Phi r. \end{aligned} \tag{3}$$

Problems remaining:

- Objective $c^T \Phi r$ is hard to evaluate.
- Number of constraints is large.

The Constraint Sampling Strategy - II

Approximation:

$$\begin{array}{lll} \text{maximize} & \tilde{c}^T \Phi \hat{r} & \\ \text{subject to} & (T\Phi r)(x) \geq (\Phi r)(x) & \text{for all } x \in \{x_1, \ldots, x_N\} \\ & r \in \mathcal{N} & \end{array} \qquad (4)$$

- $\tilde{c}^T \Phi r$ can be obtained by sampling according to the distribution c (c is positive by definition and can be made to sum to 1 without changing the problem).
- **If we sample some reasonable number of constraints, then "almost all" others will be satisfied.**
- **The constraints that are not satisfied don't distort the solution too much.**

Main Theorem - I

Given:

$$\begin{array}{ll} \text{maximize} & c^T x, \\ \text{subject to} & Ax \leq b, \end{array} \tag{5}$$

and a probability distribution μ over the rows of A.

Define $\hat{x}_N$ as the optimal solution of the following LP:

$$\begin{array}{ll} \text{maximize} & c^T x, \\ \text{subject to} & A_{i_j} x \leq b_{i_j}, \text{ for } j = 1, 2, \ldots, N, \end{array} \tag{6}$$

where A_{i_j} is the i_jth row of the matrix A, i_j are sampled IID according to a distribution μ.

Main Theorem - II

Theorem

For arbitrary $\epsilon, \delta > 0$, if $N \geq n/(\epsilon\delta) - 1$, then

$$\mathbb{P}\left\{\mu(\{i | A_i \hat{x}_N > b_i\}) \leq \epsilon\right\} \geq 1 - \delta, \quad (7)$$

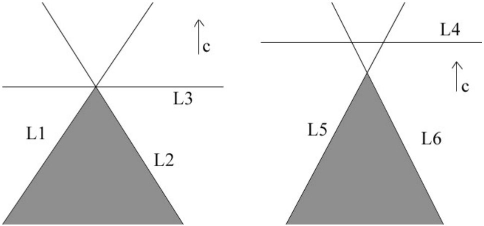
where the probability is taken over the random sampling of constraints.

- ϵ represents a tolerance or control on how many constraints are allowed to be violated.
- $1 - \delta$ represents a confidence level.
- The theorem states that given an ϵ and δ, the number of constraints we need for (7) to hold is linear in n, and does not depend on m.

Proof - I

Definition

Given an LP, a constraint is called a support constraint if the optimal objective value is changed if the constraint is relaxed.



Theorem

If there are n variables in an LP, which is bounded and feasible, then there are at most n support constraints.

Proof - II

Theorem

If $\hat{x}_N$ is the solution to the sampled LP (6)*, then*

$$\mathrm{E}\left[\mu\left(\{i : A_i \hat{x}_N > b_i\}\right)\right] \leq \frac{n}{N+1},$$

where the expectation above is taken over the random sampling of constraints.

Proof.

Considering solving problem 6 with $N+1$ constraints.

$$\mathbb{P}\left\{A_{i_{N+1}} \hat{x}_N > b_{i_{N+1}}\right\} \leq \frac{n}{N+1}.$$

It is easy to show that:

$$\mathbb{P}\left\{A_{i_{N+1}} \hat{x}_N > b_{i_{N+1}}\right\} = \mathrm{E}\left[\mu\left(\{i : A_i \hat{x}_N > b_i\}\right)\right].$$

Proof - III

From Markov inequality:

$$\mathbb{P}\left\{\mu(\{i|A_i\hat{x}_N > b_i\}) > \epsilon\right\} \leq \frac{1}{\epsilon}\mathrm{E}\left[\mu(\{i|A_i\hat{x}_N > b_i\})\right] \leq \frac{n}{\epsilon(N+1)} \leq \delta.$$

- Proof is true for any convex constraints [1]
- Proof can also be done using PAC-learning bounds of the linear classifier $x^T\tilde{a} \leq 0$ for samples $\tilde{a}$ drawn according to a fix distribution.(Vapnik-Chervonenkis [4])

How close is the solution to the relaxed problem?

Instead of finding $\tilde{r}$ that optimizes:

$$\begin{array}{lll} \text{maximize} & c^T \Phi r & \\ \text{subject to} & (T\Phi r)(x) \geq (\Phi r)(x) & \text{for all } x \in S \end{array} \tag{8}$$

We want to use $\hat{r}$ that optimizes:

$$\begin{array}{lll} \text{maximize} & c^T \Phi r & \\ \text{subject to} & (T\Phi r)(x) \geq (\Phi r)(x) & \text{for all } x \in \{x_1, \ldots, x_N\} \\ & r \in \mathcal{N} & \end{array} \tag{9}$$

where $\mathcal{N}$ is a bounded convex set which will prevent the optimization from taking too much advantage of excluded constraints.

Bound on Approximation - Theorem

Letting the constraints in problem (9) be sampled according to π_α, the "expected distribution of the initial $c^T P^t_{\mu^*}$ weighted by the value of α^t":

$$\pi_\alpha = (1-\alpha)c^T(I-\alpha P_{\mu^*})^{-1} = (1-\alpha)\sum_{t=0}^{\infty}\alpha^t c^T P^t_{\mu^*}$$

We get the following result:

Theorem

If $N \geq \frac{4K}{(1-\alpha)\epsilon\delta}\frac{\sup_{r\in\mathcal{N}}\|J^*-\Phi r\|_\infty}{c^T J^*}$ *then*

$$\|J^*-\Phi\hat{r}\|_{1,c} \leq \|J^*-\Phi\tilde{r}\|_{1,c} + \epsilon\|J^*\|_{1,c} \qquad \textit{with probability } 1-\delta$$

Bound on Approximation - Proof

$$
\begin{aligned}
\|J^* - \Phi\hat{r}\|_{1,c} &= c^T |J^* - \Phi\hat{r}| \\
&\leq c^T (I - \alpha P_{\mu^*})^{-1} |g - (I - \alpha P_{\mu^*})\Phi\hat{r}| \\
&= c^T (I - \alpha P_{\mu^*})^{-1} \big((g - (I - \alpha P_{\mu^*})\Phi\hat{r}) \\
&+ 2(g - (I - \alpha P_{\mu^*})\Phi\hat{r})^- \big) \\
&= c^T (J^* - \Phi\hat{r}) + 2c^T (I - \alpha P_{\mu^*})^{-1} (T_{\mu^*}\Phi\hat{r} - \Phi\hat{r})^- \\
&\leq c^T (J^* - \Phi\tilde{r}) + 2c^T (I - \alpha P_{\mu^*})^{-1} (T_{\mu^*}\Phi\hat{r} - \Phi\hat{r})^- \\
&\leq \|J^* - \Phi\tilde{r}\|_{1,c} + \frac{2}{1-\alpha} \pi (T_{\mu^*}\Phi\hat{r} - \Phi\hat{r})^- \\
&\leq \|J^* - \Phi\tilde{r}\|_{1,c} + \frac{2}{1-\alpha} \mu(\{i | A_i \hat{x}_N > b_i\}) \sup_{r \in \mathcal{N}} \|T\Phi r - \Phi r\|_\infty \\
&\leq \|J^* - \Phi\tilde{r}\|_{1,c} + \epsilon \|J^*\|_{1,c} \qquad \text{with probability } 1 - \delta
\end{aligned}
$$

Overall Bound on Approximation

Corollary

If $N \geq \frac{4K}{(1-\alpha)\epsilon\delta} \frac{\sup_{r\in\mathcal{N}} \|J^* - \Phi r\|_\infty}{c^T J^*}$ *and* $\Phi r = e$ *for some* r*, then:*

$$\|J^* - \Phi \hat{r}\|_{1,c} \leq \frac{2}{1-\alpha} \min_r \|\Phi r - J^*\|_\infty + \epsilon \|J^*\|_{1,c} \qquad \textit{with probability } 1-\delta$$

Remaining issues:

- Does approximating $\tilde{c}^T x$ affect the solution?
- Where to get π_α, the "expected distribution of the initial $c^T P_{\mu^*}^t$ weighted by the value of α^t"?
- How to chose the basis functions?
- $\frac{2}{1-\alpha} \min_r \|\Phi r - J^*\|_\infty$ is quite loose, can we expect better results in practice?

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