Adjustable Robust Optimization Reformulations of Two-Stage Worst-case Regret Minimization Problems

Mehran Poursoltani and Erick Delage*

Department of Decision Sciences, HEC Montréal, Montréal, Quebec, H3T 2A7, Canada

July 23, 2019

Abstract

This paper explores the idea that two-stage worst-case regret minimization problems with either objective or right-hand side uncertainty can be reformulated as two-stage robust optimization problems and can therefore benefit from the solution schemes and theoretical knowledge that have been developed in the last decade for this class of problems. In particular, we identify conditions under which a first-stage decision can be obtained either exactly using popular adjustable robust optimization decomposition schemes, or approximately by conservatively employing affine decision rules. Furthermore, we provide both numerical and theoretical evidence that in practice the first-stage decision obtained using affine decision rules is of high quality. Initially, this is done by establishing mild conditions under which these decisions can be proved exact, which effectively extends the space of regret minimization problems known to be solvable in polynomial time. We further evaluate both the sub-optimality and computational efficiency of this tractable approximation scheme in a multi-item newsvendor problem and a production transportation problem.

1 Introduction

When employing optimization in the context of uncertainty, a well-known alternative to minimizing expected value or the worst-case scenario, a.k.a. expected value model (EVM) and robust optimization (RO) respectively, consists in minimizing the regret experienced once the decision maker finds out that another action would have achieved a better performance under the realized scenario. Historically, while the paradigm of worst-case absolute regret minimization is usually attributed to Savage (1951), it became a legitimate representation of preferences through its axiomatization in Milnor (1954) and more comprehensively in Stoye (2011). Empirical studies (e.g. in Loomes and Sugden 1982 and in Bleichrodt et al. 2010) have also supported the idea that some decision makers are "regret averse" in the sense that they are inclined to abandon alternatives that might lead to large regret once they realize what would have been the best actions in hindsight. In the operations research literature, there is recently a growing number of studies that describe regret minimization models as leading to less "conservative" decisions than those produced by robust optimization (Perakis and Roels, 2008; Aissi et al., 2009; Natarajan et al., 2014; Caldentey et al., 2017). In particular, this reduced conservatism, which

*Corresponding author
Email addresses: *mehran.poursoltani@hec.ca* (Mehran Poursoltani), erick.delage@hec.ca (Erick Delage)

is often considered as the Achilles' heel of robust optimization, is achieved without requiring the assumption of knowing an underlying distribution.

An important obstacle in the application of regret minimization models resides in the fact that they can give rise to serious computational challenge. In particular, while both EVM and RO formulations are polynomially solvable in the case of a linear program with objective coefficient known to reside in their respective interval (a.k.a. box uncertainty), Averbakh and Lebedev (2005) demonstrated that solving the worst-case regret minimization form is strongly NP-hard. While there has been extensive efforts invested in the development of exact and approximate solution schemes, most of these focus on specific applications of single-stage mixed-integer programs (e.g. shortest path, knapsack, single-period portfolio optimization, etc.). More recently, some attention was driven towards general forms of two-stage continuous/mixed-integer linear programs but, to the best of our knowledge, there has been no general **tractable** conservative approximation scheme proposed for these models. In comparison, while two-stage robust optimization is also known to be strongly NP-hard when uncertainty appears in the constraints (see Guslitser 2002; Minoux 2009), there has been active research in the last 10 years about deriving and analyzing tractable solution schemes for some of the most general forms of the problem (see for instance Yanikoglu et al. 2018 for a recent survey). Moreover, these efforts have led to the development of software packages (e.g. ROME in Goh and Sim 2011 and JuMPeR in Dunning et al. 2017) that facilitate the implementation of these solution schemes and certainly promoted its use in applications. Among these different schemes, there is no doubt that the most popular one, which was initially proposed in Ben-Tal et al. (2004) and will be referred as the linear decision rule approach (as popularized in Kuhn et al. 2011), approximates the delayed decision with a decision rule that is affine with respect to the uncertain parameters.

Generally speaking, this paper explores both theoretically and numerically the idea that regret minimization problems can be reformulated as multi-stage robust optimization problems and can therefore benefit from the tractable solution schemes and theoretical knowledge that has been developed in the last decade for this class of problems. In particular, we make the following contribution:

- We establish for the first time how, in a general two-stage linear program setting with either objective or right-hand side uncertainty, both worst-case absolute regret minimization and worst-case relative regret minimization problems can be reformulated as a two-stage robust linear program. We also identify weak conditions on the regret minimization problems under which a tractable conservative approximation can be obtained by employing the concept of affine decision rules. Alternatively, we state conditions under which an exact solution can be obtained using the column-and-constraint algorithm proposed in Zeng and Zhao (2013) or in Ayoub and Poss (2016).
- We establish mild conditions on the regret minimization problem under which the theory developed in Bertsimas and Goyal (2012) and Ardestani-Jaafari and Delage (2016) can be exploited to demonstrate that the solution obtained using affine decision rules is exact. These results effectively both extend the class of regret minimization problems for which a polynomial time solution method is known to exist and support the claim that in practice affine decision rules identify solutions of high quality.
- We present the results of numerical experiments that provide further evidence that the solutions obtained using affine decision rules are of high quality. In particular, we investigate both the computational efficiency and sub-optimality of such approximate first-stage decisions in multi-item newsvendor problems and production transportation problems.

The rest of the paper is composed as follows. Section 2 reviews the relevant literature and highlights the relevance of our proposed reformulations. Section 3 summarizes some important results from the literature on two-stage robust optimization models. Section 4 proposes a two-stage robust optimization reformulation for two-stage worst-case **absolute** regret minimization with right-hand side uncertainty and objective uncertainty. Section 5 proposes a two-stage robust optimization reformulation for two-stage worst-case **relative** regret minimization with right-hand side uncertainty and objective uncertainty. Section 6 identifies conditions under which the use of affine decision rules in the robust optimization reformulations identifies exactly optimal first stage decisions. Finally, Section 7 presents our numerical experiments.

2 Literature Review

The computational challenges related to solving combinatorial worst-case regret minimization problems have been extensively tackled in the recent literature. Many of these studies investigate the complexity of specific classes of models and propose exact solution methods that exploit the underlying structure. One may refer to Kouvelis and Yu (1996) for a comprehensive review of complexity results and solution schemes established prior to 1997, and complement with Aissi et al. (2009) which provides a survey of exact and approximation solution schemes for the shortest path, spanning tree, assignment, minimum cut, minimum s-t cut, and knapsack problems. More recently, recent work has also explored applications such as a 1-median and location allocation problem on a network (Averbakh and Lebedev, 2004; Conde, 2007), a multi-market newsvendor problem (Lin and Ng, 2011), a two-machine flow shop scheduling problem (Kasperski et al., 2012), a spanning tree and a ranking problem (Audibert et al., 2014), a shortest path and subset selection problem (Natarajan et al., 2014; Pérez-Galarce et al., 2018), the knapsack problem (Furini et al., 2015), a single-machine scheduling problem with outsourcing decisions (Choi and Chung, 2016), an aircraft sequencing and scheduling problem (Ng et al., 2017), and an optimal pricing under uncertain customer valuation (Caldentey et al., 2017). Unfortunately, the numerical optimization of minimum regret problems is a real challenge in general. Just as epitomes, Averbakh (2001) studies the minimum regret version of the problem of selecting p objects, out of m, of minimal total weight when weights are only known to be within some intervals. The author identifies a polynomial algorithm yet prove that the problem becomes NP-hard when uncertainty is scenario-based or polyhedral. Averbakh and Lebedev (2004) shows that regret minimization for the minimum spanning tree and the shortest path problems with interval uncertainty about objective coefficients is generally NP-hard. Finally, Aissi et al. (2009) surveys a number of complexity results regarding classical combinatorial optimization problems. In particular, it highlights the fact that a large number of problems have the same complexity in their robust or regret minimization form if uncertainty is based on scenarios yet regret minimization typically becomes more challenging under interval uncertainty.

In the domain of continuous decision variables, most of the research has focused on the single-stage version of the problem. In particular, a small number of single stage linear regret minimization problems are known to be polynomial time solvable. As presented in Gabrel and Murat (2010), this is the case for general linear programs with right-hand side and polyhedral uncertainty since these problems can be reformulated as equivalent linear programs. Averbakh (2004) also identifies an *O(n log n)* algorithm for solving the minimum regret problem in resource allocation problems with objective and interval uncertainty. This approach is improved to linear time by Conde (2005) for the continuous knapsack problem. Nevertheless, as mentioned in the introduction, the case of a general single-stage linear program with interval objective function uncertainty is known to be strongly NP-hard (see Averbakh and Lebedev 2005) and has been the focus of much algorithmic development. First, Inuiguchi and Kume (1994) and Inuiguchi and Sakawa (1995) proposed to tackle the worst-case absolute regret minimization problem by replacing the box uncertainty set with the list of its extreme points and associated optimal hindsight actions, and inserting these points progressively using a constraint generation procedure. A similar procedure is proposed in Inuiguchi and Sakawa (1997a) where relative regret is considered. In order to speed up the identification of violated constraints, Inuiguchi and Sakawa (1996) replaces the exhaustive search with a branch and bound procedure that effectively solves a mixed integer linear programming (MILP) formulation of the regret maximization subproblem. This MILP reformulation is further improved in Mausser and Laguna (1998) by exploiting the piecewise linear structure of the regret maximization problem and a fast heuristic for identifying strong cuts is proposed in Mausser and Laguna (1999a). This constraint generation scheme is ported to the relative regret minimization problem in Mausser and Laguna (1999b). The constraint generation procedure was extended for the first time to the case of general polyhedral uncertainty appears in Inuiguchi and Sakawa (1997b) yet its low numerical efficiency later motivated the use of an outer approximation scheme (Inuiguchi et al., 1999) and cutting hyperplanes scheme (Inuiguchi and Tanino, 2001) to solve the regret maximization subproblem.

Table 1: General Approaches - Linear Single-stage Problems

Reference	Algorithm	Solution Type	Scope		
			Regret Type	Uncertain Parameters	Uncertainty Set
Inuiguchi and Kume (1994)	Constraint Generation + Vertex Enumeration	Exact	Absolute	Obj	Box
Inuiguchi and Sakawa (1995)	Constraint Generation + Vertex Enumeration	Exact	Absolute	Obj	Box
Inuiguchi and Sakawa (1996)	Constraint Generation + MILP Reformulation	Exact	Absolute	Obj	Box
Mausser and Laguna (1998)	Constraint Generation + MILP Reformulation	Exact	Absolute	Obj	Box
Mausser and Laguna (1999a)	Constraint Generation + MILP Reformulation + Greedy Search	Exact	Absolute	Obj	Box
Inuiguchi and Sakawa (1997a)	Constraint Generation + Vertex Enumeration	Exact	Relative	Obj	Box
Mausser and Laguna (1999b)	Constraint Generation + MILP Reformulation	Exact	Relative	Obj	Box
Inuiguchi and Sakawa (1997b)	Constraint Generation + MILP Reformulation	Exact	Absolute	Obj	Polyhedral
Inuiguchi et al. (1999)	Constraint Generation + Outer Approx. Scheme	Exact	Absolute	Obj	Polyhedral
Inuiguchi and Tanino (2001)	Constraint Generation + Outer Approx. Scheme + Cutting-hyperplanes scheme	Exact	Absolute	Obj	Polyhedral
Gabrel and Murat (2010)	LP Reformulation	Exact	Absolute	RHS	Box

In comparison with single-stage, the work on two-stage linear programs is rather scarce. First, in terms of application specific methods, one might consider Vairaktarakis (2000) which proposes a linear time algorithm to solve multi-item newsvendor absolute and relative regret minimization problems with interval demand uncertainty and proposes a dynamic programming approach for the NP-hard case of scenario-based uncertainty. Yue et al. (2006) and Perakis and Roels (2008) define closed form solutions for the stochastic version of this problem with only one item, absolute regret, and distribution ambiguity while Zhu et al. (2013) extends some of these results to the relative regret form. Z. (2011) also studied a related two-stage uncapacitated lot sizing problem with binary first-stage decisions and interval uncertainty on demands (i.e. right-hand sides) and identifies a dynamic programming method that is shown to provide optimal solutions in polynomial time.

Table 1 summarizes studies that propose ideas for general solution schemes. Specifically, Assavapokee et al. (2008b) considers two-stage worst-case absolute and relative regret minimization problems with binary first-stage decisions and continuous recourse variables and generalized parametric uncertainty that is based on a set of scenarios. The proposed approach is a precursor of the column-and-constraint generation (C&CG) algorithm found in Zeng and Zhao (2013) as it relies on progressively introducing worst-case scenarios (found using an exhaustive search) in a master problem that optimizes both the first-stage decisions and recourse decisions for this subset of scenarios. This C&CG approach is extended to box uncertainty set in Assavapokee et al. (2008a) where uncertainty only affects the right-hand side of constraints and the coefficients modeling the influence of the first-stage decision. This allows the authors to solve the regret maximization subproblem using two MILP reformulations that respectively generate feasibility and optimality cuts. This C&CG is further extended to polyhedral uncertainty in Jiang et al. (2013) where the subproblem is solved approximately using coordinate ascent, and in Chen et al. (2014) who successfully identifies an exact MILP reformulation when uncertainty only affects the right-hand side of constraints.

Ng (2013) investigates problems with objective functions that take the form of a sum of linearly penalized perturbed constraint violations, which are special cases of two-stage linear worst-case regret minimization problem with generalized polyhedral uncertainty. The author proposes a conservative approximation that takes the form of a two-stage robust optimization problem yet remains intractable. He employs a constraint generation scheme which involve solving a MILP at each iteration. Note that while the reformulations that we propose in sections 4 and 5 will similarly lead to two-stage robust optimization models, our reformulations will be **exact** and available whether absolute or relative regret is considered. Furthermore, by using affine decision rules, our proposed conservative approximation models will be **tractable** in the sense that they can be reformulated as linear programs of comparable size. More recently, Ning and You (2018) suggested reformulating two-stage problems with right-hand side polyhedral uncertainty exactly as two-stage robust optimization models yet did not extend this procedure to relative regret or to problems with objective uncertainty as we will present. The authors also mistakenly assume that worst-case scenarios always occur at extreme points of the polyhedral uncertainty set. This is in turn used to formulate a MILP that generates violated constraints in a C&CG approach effectively (and unknowingly) providing an optimistic approximation to the regret minimization problem (see Appendix A for an example). Finally, a distinguishing feature of our work will be to describe for the first time how linear decision rules can be tractably employed to obtain conservative solutions for a large family of two-stage regret minimization problems, and perhaps most importantly to identify conditions under which such decision rules actually return exact solutions.

Table 2: General Approaches - Two-stage Problemss

Reference	Algorithm	Solution Type	Scope			
			Regret Type	Variables (First Stage + Recourse)	Uncertain Parameters	Uncertainty Set
Assavapokee et al. (2008b)	C&CG + Exhaustive Search	Exact	Absolute Relative	Binary + Continuous	All Parameters	Discrete Scenarios
Assavapokee et al. (2008a)	C&CG + MILP Reformulation	Exact	Absolute	Binary + Continuous	RHS + First-stage Technology Matrix	Box
Jiang et al. (2013)	Constraint Generation + Coordinate Ascent	O.A.	Absolute	Binary + Continuous	RHS	Polyhedral
Ng (2013)	Constraint Generation + MILP Reformulation	C.A.	Absolute	Continuous + Continuous	RHS + Obj	Polyhedral
Chen et al. (2014)	C&CG + MILP Reformulation	Exact	Absolute	Binary + Continuous	RHS	Polyhedral
Ning and You (2018)	C&CG + MILP Reformulation	O.A.	Absolute	Continuous + Continuous	RHS	Polyhedral

* C.A. and O.A. stand for Conservative and Optimistic Approximations, respectively.

3 Modern Solution Methods for Two-stage Adjustable Robust Optimization

In this section, we present a number of modern solution methods that have appeared in recent literature concerning two-stage robust linear optimization problem. We first present the case of right-hand side uncertainty which is most relevant for the purpose of this paper. For this model, which is known to be intractable in general, the literature is rich with methods that either seek optimal solutions, conservative approximation (using affine decision rules), lower bounds, and opportunities for increased efficiency using new reformulations. We later present the case of objective uncertainty for which there is a tractable reformulation.

3.1 The Case of Fixed Recourse and Right-Hand Side Uncertainty

In this section, the focus is on the following two-stage linear robust optimization models with fixed recourse (TSLRO):

$$(TSLRO) \quad \underset{\boldsymbol{x},\boldsymbol{y}(\cdot)}{\text{maximize}} \quad \inf_{\boldsymbol{\zeta}\in\mathcal{U}} \; (C\boldsymbol{\zeta}+\boldsymbol{c})^T\boldsymbol{x}+\boldsymbol{d}^T\boldsymbol{y}(\boldsymbol{\zeta})+\boldsymbol{f}^T\boldsymbol{\zeta} \tag{1a}$$

$$\text{subject to} \quad A\boldsymbol{x}+B\boldsymbol{y}(\boldsymbol{\zeta})\leq \Psi(\boldsymbol{x})\boldsymbol{\zeta}+\boldsymbol{\psi}\,,\;\forall\,\boldsymbol{\zeta}\in\mathcal{U} \tag{1b}$$

$$\boldsymbol{x}\in\mathcal{X}\,, \tag{1c}$$

where $\boldsymbol{x}\in\mathbb{R}^{n_x}$ is the first stage decision vector implemented immediately while $\boldsymbol{y}:\mathbb{R}^{n_\zeta}\to\mathbb{R}^{n_y}$ is a strategy for the second stage decision vector that is implemented only once the vector of uncertain parameters $\boldsymbol{\zeta}\in\mathbb{R}^{n_\zeta}$ has been revealed. Furthermore, we have that $C\in\mathbb{R}^{n_x\times n_\zeta}$, $\boldsymbol{c}\in\mathbb{R}^{n_x}$, $\boldsymbol{d}\in\mathbb{R}^{n_y}$, $\boldsymbol{f}\in\mathbb{R}^{n_\zeta}$, $A\in\mathbb{R}^{m\times n_x}$ and $B\in\mathbb{R}^{m\times n_y}$, and assume that $\Psi:\mathbb{R}^{n_x}\to\mathbb{R}^{m\times n_\zeta}$ is an affine mapping of $\boldsymbol{x}$. Note that $\boldsymbol{d}$ and B are not affected by uncertainty which is also referred to as satisfying the fixed recourse property. Finally, we assume that both $\mathcal{X}$ and $\mathcal{U}$ are non-empty polyhedra such that when the latter is bounded one retrieves the more common $\min_{\boldsymbol{\zeta}\in\mathcal{U}}$ notation.

A special kind of TSLRO model emerges when the uncertain vector $\boldsymbol{\zeta}$ only influences the right-hand side of constraint (1b) and gives rise to the following definition:

Definition 1. *A TSLRO problem is considered to have "right-hand side uncertainty" when* $C=0$*,* $\boldsymbol{f}=\boldsymbol{0}$*, and* $\Psi(\boldsymbol{x})=\Psi$*.*

TSLRO problems with right-hand side uncertainty arise for instance in a number of inventory management, and logistics problems (see Melamed et al. (2016); Kim and Chung (2017); Gauvin et al. (2017); Simchi-Levi et al. (2016)).

The TSLRO problem can also be equivalently reformulated in a form where the two stages of decisions are made explicit:

$$(TSLRO)\,\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \inf_{\boldsymbol{\zeta}\in\mathcal{U}} h(\boldsymbol{x},\boldsymbol{\zeta}), \tag{2a}$$

where $h(\boldsymbol{x},\boldsymbol{\zeta})$ is defined as:

$$h(\boldsymbol{x},\boldsymbol{\zeta}) := \quad \sup_{\boldsymbol{y}} \; (C\boldsymbol{\zeta}+\boldsymbol{c})^T\boldsymbol{x}+\boldsymbol{d}^T\boldsymbol{y}+\boldsymbol{f}^T\boldsymbol{\zeta} \tag{3a}$$

$$\text{s.t.} \;\; A\boldsymbol{x}+B\boldsymbol{y}\leq\Psi(\boldsymbol{x})\boldsymbol{\zeta}+\boldsymbol{\psi}\,. \tag{3b}$$

In Ben-Tal et al. (2004), the authors established that the TSLRO problem is NP-hard in general due to the so-called "adversarial problem", i.e. $\inf_{\boldsymbol{\zeta}\in\mathcal{U}} h(\boldsymbol{x},\boldsymbol{\zeta})$, which reduces to the minimization of a piecewise linear concave function over an arbitrary polyhedron. Since this seminal work, a number of methods have been proposed to circumvent this issue. We present a subset of these methods in the rest of this section where it will be useful to refer to some of the following assumptions.

Assumption 1. *The sets $\mathcal{X}$ and $\mathcal{U}$ are non-empty polyhedra of the respective form $\mathcal{X} := \{\boldsymbol{x} \in \mathbb{R}^{n_x} | W\boldsymbol{x} \leq \boldsymbol{v}\}$, with $W \in \mathbb{R}^{r \times n_x}$ and $\boldsymbol{v} \in \mathbb{R}^r$, and $\mathcal{U} := \{\boldsymbol{\zeta} \in \mathbb{R}^{n_\zeta} | P\boldsymbol{\zeta} \leq \boldsymbol{q}\}$, with $P \in \mathbb{R}^{s \times n_\zeta}$ and $\boldsymbol{q} \in \mathbb{R}^s$. Furthermore, there exists a triplet $(\boldsymbol{x}, \boldsymbol{\zeta}, \boldsymbol{y})$ such that $\boldsymbol{x} \in \mathcal{X}$, $\boldsymbol{\zeta} \in \mathcal{U}$, and $A\boldsymbol{x} + B\boldsymbol{y} \leq \Psi(\boldsymbol{x})\boldsymbol{\zeta} + \boldsymbol{\psi}$.*

Assumption 2. *The feasible set $\mathcal{X}$ is such that it is always possible to identify a recourse action $\boldsymbol{y}$ that will satisfy all the constraints under any realization $\boldsymbol{\zeta} \in \mathcal{U}$, a property commonly referred as "relatively complete recourse". Specifically:*

$$\mathcal{X} \subseteq \{\boldsymbol{x} \in \mathbb{R}^{n_x} | \forall \boldsymbol{\zeta} \in \mathcal{U}, \exists \boldsymbol{y} \in \mathbb{R}^{n_y}, A\boldsymbol{x} + B\boldsymbol{y} \leq \Psi(\boldsymbol{x})\boldsymbol{\zeta} + \boldsymbol{\psi}\}\,. \tag{4}$$

Assumption 3. *For all $\boldsymbol{x} \in \mathcal{X}$ there exists a $\boldsymbol{\zeta} \in \mathcal{U}$ such that the recourse problem (3) is bounded. In other words, this assumes that the TSLRO problem is bounded.*

3.1.1 The Column-and-Constraint Generation Method

A so-called column-and-constraint generation (C&CG) method was proposed in Zeng and Zhao (2013) to identify an exact solution for the TSLRO problem. Specifically, in its simplest form this method can be applied when assumptions 1, 2, and 3 are satisfied together with the following assumption.

Assumption 4. *For all feasible first stage decision, there is a lower bound on the worst-case profit achievable, i.e. for all $\boldsymbol{x} \in \mathcal{X}$, $\inf_{\boldsymbol{\zeta} \in \mathcal{U}} h(\boldsymbol{x}, \boldsymbol{\zeta}) > -\infty$.*

In particular, the latter assumption is straightforwardly met when the uncertainty set $\mathcal{U}$ is bounded. The C&CG method then exploits the fact that $h(\boldsymbol{x}, \boldsymbol{\zeta})$ is convex with respect to $\boldsymbol{\zeta}$ to reformulate problem (2) equivalently as :

$$\underset{\boldsymbol{x} \in \mathcal{X}}{\text{maximize}} \quad \min_{\boldsymbol{\zeta} \in \mathcal{U}_v} h(\boldsymbol{x}, \boldsymbol{\zeta})\,,$$

where $\mathcal{U}_v = \{\bar{\boldsymbol{\zeta}}_1, \bar{\boldsymbol{\zeta}}_2, ..., \bar{\boldsymbol{\zeta}}_K\}$ is the set of vertices of $\mathcal{U}$, i.e. $\mathcal{U} = \text{ConvexHull}(\mathcal{U}_v)$ when $\mathcal{U}$ is bounded. This allows one to decompose the TSLRO problem as a restricted master problem:

$$\begin{aligned}
&\underset{\boldsymbol{x}, \{\boldsymbol{y}_k\}_{k=1}^k}{\text{maximize}} && \min_{k \in \mathcal{K}} \boldsymbol{c}(\bar{\boldsymbol{\zeta}}_k)^T \boldsymbol{x} + \boldsymbol{d}^T \boldsymbol{y}_k + \boldsymbol{f}^T \bar{\boldsymbol{\zeta}}_k && \text{(5a)}\\
&\text{subject to} && A\boldsymbol{x} + B\boldsymbol{y}_k \leq \Psi(\boldsymbol{x})\bar{\zeta}_k + \boldsymbol{\psi}\,,\, \forall\, k \in \mathcal{K}' && \text{(5b)}\\
& && \boldsymbol{x} \in \mathcal{X}\,, && \text{(5c)}
\end{aligned}$$

where $\mathcal{K}' \subseteq \{1, 2, \ldots, K\}$ and each $\boldsymbol{y}_k \in \mathbb{R}^{n_y}$ which provides an upper bound for the optimal value of the TSLRO problem. This bound can be further tighten by introducing additional vertices in $\mathcal{K}'$. Given any $\boldsymbol{x} \in \mathcal{X}$ which is optimal with respect to problem (5), one can identify an additional worst-case vertex by solving the NP-hard adversarial problem $\min_{\boldsymbol{\zeta} \in \mathcal{U}_v} h(\boldsymbol{x}, \boldsymbol{\zeta})$. Recently, it has become common practice to reformulate this problem as the following mixed-integer linear program:

$$\begin{aligned}
&\underset{\boldsymbol{\zeta} \in \mathcal{U}, \boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{u}}{\text{minimize}} && \boldsymbol{x}^T C \boldsymbol{\zeta} + \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{d}^T \boldsymbol{y} + \boldsymbol{f}^T \boldsymbol{\zeta} && \text{(6a)}\\
&\text{subject to} && A\boldsymbol{x} + B\boldsymbol{y} \leq \Psi(\boldsymbol{x})\boldsymbol{\zeta} + \boldsymbol{\psi} && \text{(6b)}\\
& && \boldsymbol{\lambda} \geq 0 && \text{(6c)}\\
& && \boldsymbol{\lambda} \leq M\boldsymbol{u} && \text{(6d)}\\
& && \Psi(\boldsymbol{x})\boldsymbol{\zeta} + \boldsymbol{\psi} - A\boldsymbol{x} - B\boldsymbol{y} \leq M(1 - \boldsymbol{u}) && \text{(6e)}\\
& && \boldsymbol{d} = B^T \boldsymbol{\lambda} && \text{(6f)}\\
& && \boldsymbol{u} \in \{0, 1\}^m\,, && \text{(6g)}
\end{aligned}$$

where $\boldsymbol{y} \in \mathbb{R}^{n_y}$, $\boldsymbol{\lambda} \in \mathbb{R}^m$, and M is some large enough constant.

3.1.2 Conservative Approximation using Linear Decision Rules

A common approach (initially proposed in Ben-Tal et al. 2004) to formulate a tractable approximation for the TSLRO problem consists in restricting $\boldsymbol{y}(\cdot)$ to take the form of an affine policy $\boldsymbol{y}(\boldsymbol{\zeta}) := Y\boldsymbol{\zeta} + \boldsymbol{y}$, where $Y \in \mathbb{R}^{n_y \times n_\zeta}$ and $\boldsymbol{y} \in \mathbb{R}^{n_y}$. This gives rise to what is commonly referred as the affinely adjustable robust counterpart (AARC) model:

$$(AARC) \quad \underset{\boldsymbol{x}\in\mathcal{X},\boldsymbol{y},Y}{\text{maximize}} \quad \inf_{\boldsymbol{\zeta}\in\mathcal{U}} (C\boldsymbol{\zeta} + \boldsymbol{c})^T\boldsymbol{x} + \boldsymbol{d}^T(Y\boldsymbol{\zeta} + \boldsymbol{y}) + \boldsymbol{f}^T\boldsymbol{\zeta} \tag{7a}$$

$$\text{subject to} \quad A\boldsymbol{x} + B(Y\boldsymbol{\zeta} + \boldsymbol{y}) \leq \Psi(\boldsymbol{x})\boldsymbol{\zeta} + \boldsymbol{\psi}\,, \; \forall\, \boldsymbol{\zeta} \in \mathcal{U}\,. \tag{7b}$$

It is said that the AARC problem conservatively approximates the TSLRO problem since it identifies a solution pair $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}(\cdot))$ that is necessarily feasible according to the TSLRO model and since its optimal value provides a lower bound on the optimal value of the TSLRO problem.

A linear programming reformulation of problem (7) can be obtained by exploiting Assumption 1 which ensures that $\mathcal{U}$ is non-empty together with the principles of duality theory. Indeed, each minimization problem in $\boldsymbol{\zeta}$ that appears in the evaluation of the objective function or of each constraint is a feasible linear program. Strong duality therefore holds and can be exploited to derive problem (7)'s so called robust counterpart:

$$\underset{\boldsymbol{x}\in\mathcal{X},\boldsymbol{y},Y,\Lambda,\boldsymbol{\lambda}}{\text{maximize}} \quad \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} - \boldsymbol{q}^T\boldsymbol{\lambda} \tag{8a}$$

$$\text{subject to} \quad C^T\boldsymbol{x} + Y^T\boldsymbol{d} + \boldsymbol{f} + P^T\boldsymbol{\lambda} = 0 \tag{8b}$$

$$A\boldsymbol{x} + B\boldsymbol{y} - \boldsymbol{\psi} + \Lambda\boldsymbol{q} \leq 0 \tag{8c}$$

$$\Psi(\boldsymbol{x}) - BY + \Lambda P = 0 \tag{8d}$$

$$\Lambda \geq 0, \boldsymbol{\lambda} \geq 0\,, \tag{8e}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^s$ and $\Lambda \in \mathbb{R}^{m\times s}$ are the dual variables that arise when applying duality to the objective function (7a) and each constraint of (7b), respectively.

In the last decade, a number of theoretical and empirical arguments have reinforced a prevailing belief that linear decision rules provide high quality solutions to TSLRO problems. One might for instance refer to B. et al. (2010) and Ardestani-Jaafari and Delage (2016) for conditions under which this approach is exact. A wide range of applications have also benefited from this approach including portfolio selection (Chen et al., 2011), inventory management (Ben-Tal et al., 2005), and energy planning (Bertsimas et al., 2013), etc.

3.1.3 Other Solution Schemes

There exists a rich pool of additional methods that have been proposed to solve TSLRO problems of the form presented in (1). While we encourage the reader to refer to Delage and Iancu (2015) and Yanikoglu et al. (2018) for a more exhaustive description, we summarize below the main categories of approach.

In terms of exact methods, it is worth mentioning the work of Ayoub and Poss (2016) which provide an alternate column-and-constraint generation algorithm for deriving the exact solutions of TSLRO problem where $C = 0$ and $\boldsymbol{d} = 0$ so that $\boldsymbol{\zeta}$ only appears in the constraints which makes it particularly attractive for problems where Assumption 2 is violated.

In terms of approximation methods, Kuhn et al. (2011) shows how linear decision rules can also be applied on a dual maximization problem associated to the TSLRO to obtain lower bounds on its optimal value. Alternatively, one can also obtain lower bounds by replacing $\mathcal{U}$ with a finite subset of carefully selected scenarios (see Hadjiyiannis et al. 2011). Regarding conservative approximations, the work of Chen et al. (2008) and Chen and Zhang (2009) further explains how piecewise linear (a.k.a. segregated) linear decision rules can be used instead of plain linear ones, while Ben-Tal et al. (2009) and Bertsimas et al. (2011) investigated the use of quadratic and polynomial decision rules respectively. These results are part of a long list of attempts of finding tractable conservative approximation models that can close the optimality gap associated to linear decision rules. We refer the reader to Table 3 in Yanikoglu et al. (2018) for a more detailed list of decision rule approximation schemes. Additional schemes have been proposed to improve the quality of solutions obtained from decision rule approaches. For instance, in Zhen et al. (2018), the authors propose exploiting Fourier?Motzkin elimination to eliminate a subset of adjustable variables before replacing the remaining adjustable variables with a structured decision rule. Alternatively, Ardestani-Jaafari and Delage (2017) proposes reformulating second-stage problems that satisfy Assumption 2 as an equivalent "complete recourse" problem which introduce additional adjustable variables prior to exploiting structured decision rules in order to tighten the bound obtain from a certain parametric family of rules.

Interestingly, it was recently observed in Bertsimas and de Ruiter (2016) that any TSLRO problem could be equivalently reformulated as an equivalent "dualized" TSLRO problem where the worst-case optimization problem is formulated in terms of the dual variables of the second-stage problem instead of $\boldsymbol{\zeta}$. The authors provide numerical evidence that this can in practice be beneficial in terms of numerical efficiency when using affine decision rules. This also allows them to propose models that identify tighter lower bounds on TSLRO by exploiting the idea of Hadjiyiannis et al. (2011) on both versions of the TSLRO. Similarly, one might suspect that exact methods such as C&CG might perform differently whether they are applied on the original TSLRO or its dualized form.

Finally, an important recent methodological development consists in deriving exact copositve programming reformulations for the TSLRO problem (see Xu and Burer 2018 and Hanasusanto and Kuhn 2018). While copositive programming is known to be NP-hard in general, there are known hierarchies of tractable approximation models for these mathematical programs that will eventually identify an exactly optimal solutions. This gives rise to new conservative approximation models, typically of the form of semidefinite programs, that can be designed in ways that ensures that the solutions are at least as good as the solutions obtained from affine decision rules (see Appendix G in Ardestani-Jaafari and Delage 2017).

3.2 The Case of Objective Function Uncertainty

An alternative class of two-stage robust linear optimization problems makes the assumption that the uncertainty is limited to the objective function. This is summarized in the following formulation:

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathcal{X},\boldsymbol{y}(\cdot)}{\text{maximize}} \quad & \inf_{\boldsymbol{\zeta}\in\mathcal{U}} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T(\boldsymbol{\zeta})\boldsymbol{y}(\boldsymbol{\zeta}) && \text{(9a)} \\ \text{subject to} \quad & A\boldsymbol{x} + B\boldsymbol{y}(\boldsymbol{\zeta}) \leq \boldsymbol{\psi}\,,\ \forall\,\boldsymbol{\zeta}\in\mathcal{U}\,, && \text{(9b)} \end{aligned}$$

where $\boldsymbol{d} : \mathbb{R}^{n_\zeta} \to \mathbb{R}^{n_y}$ is assumed to be an affine mapping of $\boldsymbol{\zeta}$, *i.e.,* we can characterize it in the form $\boldsymbol{d}(\boldsymbol{\zeta}) := D\boldsymbol{\zeta} + \boldsymbol{d}$, for some $D \in \mathbb{R}^{n_y \times n_\zeta}$ and $\boldsymbol{d} \in \mathbb{R}^{n_y}$. As for the case of TSLRO, the model can be reformulated in a format that emphasizes the dynamics:

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \inf_{\boldsymbol{\zeta}\in\mathcal{U}} \ h(\boldsymbol{x},\boldsymbol{\zeta})\,, \tag{10a}$$

where the recourse problem is defined as:

$$\begin{aligned} h(\boldsymbol{x},\boldsymbol{\zeta}) := \quad & \sup_{\boldsymbol{y}} \ \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T(\boldsymbol{\zeta})\boldsymbol{y} && \text{(11a)} \\ & \text{s.t.} \ \ A\boldsymbol{x} + B\boldsymbol{y} \leq \boldsymbol{\psi}\,. && \text{(11b)} \end{aligned}$$

From a computational perspective, it is interesting to consider the case where assumptions 1, 2 and 3 are applicable. In particular, Assumption 2, which was referred as relatively complete recourse, simply reduces to the fact that $\mathcal{X} \subseteq \{\boldsymbol{x} \in \mathbb{R}^{n_x} | \exists \boldsymbol{y} \in \mathbb{R}^{n_y}, A\boldsymbol{x} + B\boldsymbol{y} \leq \boldsymbol{\psi}\}$. Under these conditions, problem (9) becomes more appealing than the TSLRO problem in (1) as one can easily verify that it can be reformulated as an equivalent linear program.

Proposition 1. *Given that assumptions 1, 2 and 3 are satisfied, problem* (9) *can be reformulated as the following equivalent linear program:*

$$\begin{aligned}
\underset{\boldsymbol{x}\in\mathcal{X},\boldsymbol{y}',\boldsymbol{\lambda}}{\text{maximize}} \quad & \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} - \boldsymbol{q}^T\boldsymbol{\lambda} && (12a)\\
\text{subject to} \quad & A\boldsymbol{x} + B\boldsymbol{y} \leq \boldsymbol{\psi} && (12b)\\
& P^T\boldsymbol{\lambda} + D^T\boldsymbol{y} = 0 && (12c)\\
& \boldsymbol{\lambda} \geq 0\,, && (12d)
\end{aligned}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^s$.

Proof. Based on Assumption 2, for all $\boldsymbol{x} \in \mathcal{X}$ and all $\boldsymbol{\zeta} \in \mathcal{U}$, there exists a $\boldsymbol{y}$ for which problem (11) is feasible. Therefore, strong duality property holds for problem (11) and duality can be used to reformulate it as a minimization problem:

$$\begin{aligned}
h(\boldsymbol{x},\boldsymbol{\zeta}) := \inf_{\boldsymbol{\rho}} \quad & \boldsymbol{c}^T\boldsymbol{x} + (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} && (13a)\\
\text{s.t.} \quad & B^T\boldsymbol{\rho} = \boldsymbol{d}(\boldsymbol{\zeta}) && (13b)\\
& \boldsymbol{\rho} \geq 0\,, && (13c)
\end{aligned}$$

where $\boldsymbol{\rho} \in \mathbb{R}^m$ is the dual variable associated to constraint (11b). Therefore, the adversarial problem (10) can be rewritten as problem (14):

$$\begin{aligned}
\inf_{\boldsymbol{\zeta}\in\mathcal{U}} h(\boldsymbol{x},\boldsymbol{\zeta}) = \quad \inf_{\boldsymbol{\zeta},\boldsymbol{\rho}} \quad & \boldsymbol{c}^T\boldsymbol{x} + (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} && (14a)\\
\text{s.t.} \quad & B^T\boldsymbol{\rho} = D\boldsymbol{\zeta} + \boldsymbol{d} && (14b)\\
& \boldsymbol{\rho} \geq 0 && (14c)\\
& P\boldsymbol{\zeta} \leq \boldsymbol{q}\,, && (14d)
\end{aligned}$$

where we exploited the definition of $\boldsymbol{d}(\boldsymbol{\zeta})$.

According to Assumption 3, for all $\boldsymbol{x} \in \mathcal{X}$ there is a $\hat{\boldsymbol{\zeta}} \in \mathcal{U}$ for which problem (11) is bounded, and it has a finite optimal value based on Assumption 2. By strong duality property , problem (13) must also have a finite optimal value for the same $\hat{\boldsymbol{\zeta}}$, hence it must have a feasible solution $\hat{\boldsymbol{\rho}}$. We conclude that $(\hat{\boldsymbol{\zeta}},\ \hat{\boldsymbol{\rho}})$ is a feasible solution for problem (13). Therefore, strong duality applies for the minimization problem in (14) and ensures that

$$\begin{aligned}
\inf_{\boldsymbol{\zeta}\in\mathcal{U}} h(\boldsymbol{x},\boldsymbol{\zeta}) = \quad \sup_{\boldsymbol{y}',\boldsymbol{\lambda},\boldsymbol{\gamma}} \quad & \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y}' - \boldsymbol{q}^T\boldsymbol{\lambda}\\
\text{s.t.} \quad & A\boldsymbol{x} + B\boldsymbol{y}' + \boldsymbol{\gamma} = \boldsymbol{\psi}\\
& P^T\boldsymbol{\lambda} + D^T\boldsymbol{y}' = 0\\
& \boldsymbol{\gamma} \geq 0, \boldsymbol{\lambda} \geq 0\,,
\end{aligned}$$

where $\boldsymbol{y} \in \mathbb{R}^{n_y}$, $\boldsymbol{\gamma} \in \mathbb{R}^m$ and $\boldsymbol{\lambda} \in \mathbb{R}^s$ are the dual variables associated with the constraints (14b), (14c), and (14d) respectively. This maximization problem can be reintegrated with the maximization over $\boldsymbol{x} \in \mathcal{X}$ to obtain problem (12). □

4 TSLRO Reformulations for Worst-case Absolute Regret Minimization Problems

As defined in Savage (1951), the worst-case absolute regret criterion aims at evaluating the performance of a decision $\boldsymbol{x}$ with respect to the so-called "worst-case regret" that might be experienced in hindsight when comparing $\boldsymbol{x}$ to the best decision that could have been made. Mathematically speaking, given a profit function $h(\boldsymbol{x}, \boldsymbol{\zeta})$ which depends on both the decision and the realization of some uncertain vector of parameters $\boldsymbol{\zeta}$, one measures the regret experienced once $\boldsymbol{\zeta}$ is revealed as the difference between the best profit achievable $\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})$ and the profit $h(\boldsymbol{x}, \boldsymbol{\zeta})$ achieved by the decision $\boldsymbol{x}$ that was implemented. The worst-case absolute regret minimization (WCARM) problem thus takes the form:

$$\text{(WCARM)} \quad \underset{\boldsymbol{x} \in \mathcal{X}}{\text{minimize}} \; \sup_{\boldsymbol{\zeta} \in \mathcal{U}} \left\{ \sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) - h(\boldsymbol{x}, \boldsymbol{\zeta}) \right\}, \tag{15}$$

which is well defined when one makes the assumption that the best profit achievable in hindsight never reaches infinity under any scenario for $\boldsymbol{\zeta}$.

Assumption 5. *The best profit achievable is bounded, i.e.,* $\sup_{\boldsymbol{\zeta} \in \mathcal{U}, \boldsymbol{x} \in \mathcal{X}} h(\boldsymbol{x}, \boldsymbol{\zeta}) < \infty$.

Assumption 5 is a natural condition to impose on the WCARM problem and implies Assumption 3. When Assumption 5 is not known to be satisfied, we will interpret the WCARM model as:

$$\underset{\boldsymbol{x} \in \mathcal{X}}{\text{minimize}} \; \sup_{\boldsymbol{\zeta} \in \mathcal{U}} \left\{ \sup_{\boldsymbol{x}' \in \mathcal{X}, \boldsymbol{y}' \in \mathcal{Y}(\boldsymbol{x}')} \inf_{\boldsymbol{y} \in \mathcal{Y}(\boldsymbol{x})} \boldsymbol{c}^T (\boldsymbol{x}' - \boldsymbol{x}) + \boldsymbol{d}^T (\boldsymbol{y}' - \boldsymbol{y}) \right\}$$

and interpret the fact that WCARM is unbounded as indicating that the worst-case absolute regret is zero since there exists an $\boldsymbol{x} \in \mathcal{X}$ such that for all $\boldsymbol{\zeta} \in \mathcal{U}$ there is a way of reaching an arbitrarily large profit.[1]

While we encourage interested readers to read an extensive review of the recent work regarding this problem formulation in Aissi et al. (2009), in what follows we demonstrate how the WCARM problem can be reformulated as a TSLRO when the profit function $h(\boldsymbol{x}, \boldsymbol{\zeta})$ captures the profit of a second-stage linear decision model with either right-hand side or objective uncertainty.

4.1 The Case of Right-Hand Side Uncertainty

We consider the case where $h(\boldsymbol{x}, \boldsymbol{\zeta})$ takes the form presented in problem (3) and where uncertainty is limited to the right-hand side as defined in Definition 1.

Proposition 2. *Given that Assumption 1 is satisfied, the WCARM problem with right-hand side uncertainty is equivalent to the following TSLRO problem:*

$$\underset{\boldsymbol{x} \in \mathcal{X}, \boldsymbol{y}'(\cdot)}{\text{maximize}} \quad \inf_{\boldsymbol{\zeta}' \in \mathcal{U}'} \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{d}^T \boldsymbol{y}'(\boldsymbol{\zeta}') + \boldsymbol{f}'^T \boldsymbol{\zeta}' \tag{16a}$$

$$\text{subject to} \quad A\boldsymbol{x} + B\boldsymbol{y}'(\boldsymbol{\zeta}') \leq \Psi' \boldsymbol{\zeta}' + \boldsymbol{\psi}, \; \forall \boldsymbol{\zeta}' \in \mathcal{U}', \tag{16b}$$

where $\boldsymbol{\zeta}' \in \mathbb{R}^{n_\zeta + n_x + n_y}$, $\boldsymbol{y}' : \mathbb{R}^{n_\zeta + n_x + n_y} \to \mathbb{R}^{n_y}$, $\boldsymbol{f}' = [\boldsymbol{0}^T \;\; -\boldsymbol{c}^T \;\; -\boldsymbol{d}^T]^T$, *and* $\Psi' := \begin{bmatrix} \Psi & 0 & 0 \end{bmatrix}$, *while* $\mathcal{U}'$ *is defined as a new uncertainty set as:*

$$\mathcal{U}' := \{\boldsymbol{\zeta}' \in \mathbb{R}^{n_\zeta + n_x + n_y} | P' \boldsymbol{\zeta}' \leq \boldsymbol{q}'\} \tag{17}$$

with

$$P' = \begin{bmatrix} P & 0 & 0 \\ 0 & W & 0 \\ -\Psi & A & B \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{q}' := \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{v} \\ \boldsymbol{\psi} \end{bmatrix}.$$

Furthermore, this TSLRO reformulation naturally satisfies Assumption 1, but also satisfies assumptions 2 and 3 if the WCARM problem satisfies assumptions 2 and assumptions 2 and3 respectively, and satisfies Assumption 4 if the WCARM problem satisfies assumptions 4 and 5.

[1] Note that if WCARM is unbounded it is necessarily because such an $\boldsymbol{x} \in \mathcal{X}$ exists since for any fixed $\boldsymbol{x}$ if the profit reachable under all $\boldsymbol{\zeta} \in \mathcal{U}$ is finite then the regret is necessarily non-negative.

Proof. By substituting problem (3) in problem (15) after replacing $C = 0$, $\boldsymbol{f} = \boldsymbol{0}$, and $\Psi(\boldsymbol{x}) = \Psi$ as prescribed by Definition 1, we can proceed with the following simple steps:

$$\begin{aligned}
\text{WCARM} \equiv \underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \quad & \sup_{\boldsymbol{\zeta}\in\mathcal{U}} \left\{ \sup_{\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta})} \boldsymbol{c}^T\boldsymbol{x}' + \boldsymbol{d}^T\boldsymbol{y}' - \sup_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} \right\} && \text{(18a)}\\
\equiv \underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \quad & \sup_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta})} \; \inf_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} \boldsymbol{c}^T\boldsymbol{x}' + \boldsymbol{d}^T\boldsymbol{y}' - \boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{d}^T\boldsymbol{y} && \text{(18b)}\\
\equiv \underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad & \inf_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta})} \; \sup_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} -\boldsymbol{c}^T\boldsymbol{x}' - \boldsymbol{d}^T\boldsymbol{y}' + \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y}\,, && \text{(18c)}
\end{aligned}$$

where $\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta}) := \{\boldsymbol{y} \in \mathbb{R}^{n_y} | A\boldsymbol{x} + B\boldsymbol{y} \leq \Psi\boldsymbol{\zeta} + \boldsymbol{\psi}\}$, and where we simply regrouped the minimization and maximization operations together, and later rewrote the minimization problem as a maximization problem with the understanding that an optimal value for WCARM can be obtained by changing the sign of the optimal value returned from problem (18c).

In order to formulate a TSLRO model, we simply consider a lifted uncertain vector composed as $\boldsymbol{\zeta}' := [\boldsymbol{\zeta}^T\ \boldsymbol{x}'^T\ \boldsymbol{y}'^T]^T$ which needs to realize inside the polyhedron defined as

$$\mathcal{U}' := \{[\boldsymbol{\zeta}^T\ \boldsymbol{x}^T\ \boldsymbol{y}^T]^T \in \mathbb{R}^{n_\zeta+n_x+n_y} \,|\, P\boldsymbol{\zeta} \leq \boldsymbol{q},\ \boldsymbol{x} \in \mathcal{X},\ A\boldsymbol{x} + B\boldsymbol{y} \leq \Psi\boldsymbol{\zeta} + \boldsymbol{\psi}\}\,.$$

One also needs to consider that since $\boldsymbol{\zeta}$ has been lifted to $\boldsymbol{\zeta}'$, the recourse decision $\boldsymbol{y}$ can depend on all the information revealed by $\boldsymbol{\zeta}'$. This completes the proof of how the TSLRO model presented in (16) is equivalent to the WCARM.

As for verifying the conditions under which all four assumptions are satisfied by this new TSLRO, one can start by confirming that if Assumption 1 is satisfied for the WCARM problem, then there exists a triplet $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{y}})$ that are such that $\bar{\boldsymbol{x}} \in \mathcal{X}$, $\bar{\boldsymbol{\zeta}} \in \mathcal{U}$, and $\bar{\boldsymbol{y}} \in \mathcal{Y}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\zeta}})$. It is then not difficult to confirm that $\boldsymbol{\zeta}' := [\bar{\boldsymbol{\zeta}}^T\ \bar{\boldsymbol{x}}^T\ \bar{\boldsymbol{y}}^T]^T$ must be a member of $\mathcal{U}'$ so that the triplet $(\bar{\boldsymbol{x}}, \boldsymbol{\zeta}', \bar{\boldsymbol{y}})$ satisfies the same condition for the new TSLRO problem (16) hence Assumption 1 applies. Secondly, given that the feasible set for the recourse problem is the same in WCARM and its new TSLRO reformulation, Assumption 2 carries over to the new TSLRO problem. Thirdly, Assumption 3 also carries through since given any policy $\bar{\boldsymbol{\zeta}} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_\zeta}$ which verifies that the assumption holds for the WCARM, one can construct a policy $\boldsymbol{\zeta}'(\boldsymbol{x}) := [\bar{\boldsymbol{\zeta}}(\boldsymbol{x})^T\boldsymbol{x}'^T\boldsymbol{y}'^T]^T$, with any feasible pair $(\boldsymbol{x}', \boldsymbol{y}')$ provided using Assumption 2, that will make the condition hold in the TSLRO. Finally, Assumption 4 carries through to the new TSLRO as long as the WCARM also satisfies assumptions 5. Indeed, when both assumptions are satisfied by the WCARM problem, we know that:

$$\begin{aligned}
\inf_{\boldsymbol{\zeta}'\in\mathcal{U}'} h'(\boldsymbol{x}, \boldsymbol{\zeta}') &= \inf_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta})} \; \sup_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} -\boldsymbol{c}^T\boldsymbol{x}' - \boldsymbol{d}^T\boldsymbol{y}' + \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y}\\
&\geq \inf_{\boldsymbol{\zeta}\in\mathcal{U}} h(\boldsymbol{x}, \boldsymbol{\zeta}) - \sup_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta})} \boldsymbol{c}^T\boldsymbol{x}' + \boldsymbol{d}^T\boldsymbol{y}'\\
&\geq \inf_{\boldsymbol{\zeta}\in\mathcal{U}} h(\boldsymbol{x}, \boldsymbol{\zeta}) - \sup_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}, \boldsymbol{\zeta}) > -\infty\,.
\end{aligned}$$

□

Proposition 2 states that the WCARM model with right-hand side uncertainty can be reformulated as a TSLRO problem. This is particularly interesting since it implies that it can benefit from the exact solution methods and conservative approximations discussed in sections 3.1.1, 3.1.2, and 3.1.3. As an example, we provide below how the affine decision rules, introduced in Section 3.1.2, can be applied to this reformulation

Corollary 3. *Given that Assumption 1 is satisfied, the WCARM problem with right-hand side uncertainty is conservatively approximated by*

$$\begin{aligned}
\underset{\boldsymbol{x}\in\mathcal{X},\boldsymbol{y},Y',\Lambda',\boldsymbol{\lambda}'}{\text{minimize}} \quad & -\boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{d}^T\boldsymbol{y} + \boldsymbol{q}'^T\boldsymbol{\lambda}' && \text{(19a)}\\
\text{subject to} \quad & -\boldsymbol{c}' - \boldsymbol{d}' + Y'^T\boldsymbol{d} + P'^T\boldsymbol{\lambda}' = 0 && \text{(19b)}\\
& A\boldsymbol{x} + B\boldsymbol{y} - \boldsymbol{\psi} + \Lambda'\boldsymbol{q}' \leq 0 && \text{(19c)}\\
& \Psi' - BY' + \Lambda'P' = 0 && \text{(19d)}\\
& \Lambda' \geq 0, \boldsymbol{\lambda}' \geq 0\,, && \text{(19e)}
\end{aligned}$$

where $Y' \in \mathbb{R}^{n_y \times n_\zeta + n_x + n_y}$*,* $\Lambda' \in \mathbb{R}^{m \times s+r+m}$*, and* $\boldsymbol{\lambda}' \in \mathbb{R}^{s+r+m}$*.*

It is worth noting that to obtain the reformulation presented in Corollary 3, one needs to employ decision rules of the form $\boldsymbol{y}'(\boldsymbol{\zeta}') := Y'\boldsymbol{\zeta}' + \boldsymbol{y} = Y_\zeta\boldsymbol{\zeta} + Y_x\boldsymbol{x}' + Y_y\boldsymbol{y}' + \boldsymbol{y}$, for some $Y_\zeta \in \mathbb{R}^{n_y \times n_\zeta}$, $Y_x \in \mathbb{R}^{n_y \times n_x}$, and $Y_y \in \mathbb{R}^{n_y \times n_y}$, and where $(\boldsymbol{x}', \boldsymbol{y}')$ captures the best pair of actions one would have implemented if he had a-priori information about $\boldsymbol{\zeta}$.

If one is more interested in applying an exact method for solving WCARM, then as long as the WCARM problem satisfies assumptions 1, 2, 3, 4, and 5, based on Proposition 2 one can straightforwardly apply the column-and-constraint generation algorithm proposed in Section 3.1.1 to the TRSLO problem (16).

4.2 The Case of Objective Uncertainty

We consider the case where $h(\boldsymbol{x}, \boldsymbol{\zeta})$ takes the form presented in problem (11).

Proposition 4. *Given that assumptions 1 and 2 are satisfied, the WCARM problem with objective uncertainty is equivalent to the following TSLRO problem:*

$$\begin{aligned}
\underset{\boldsymbol{x},\boldsymbol{y}'(\boldsymbol{\zeta})}{\text{maximize}} \quad & \inf_{\boldsymbol{\zeta}'\in\mathcal{U}'} (C'\boldsymbol{\zeta}' + \boldsymbol{c})^T\boldsymbol{x} + \boldsymbol{d}'^T\boldsymbol{y}'(\boldsymbol{\zeta}') + \boldsymbol{f}'^T\boldsymbol{\zeta}' && \text{(20a)}\\
\text{subject to} \quad & A'\boldsymbol{x} + B'\boldsymbol{y}'(\boldsymbol{\zeta}') \leq \Psi'\boldsymbol{\zeta}' + \boldsymbol{\psi}' && \text{(20b)}\\
& \boldsymbol{x} \in \mathcal{X}, && \text{(20c)}
\end{aligned}$$

where $\boldsymbol{y}' : \mathbb{R}^{n_\zeta + m} \to \mathbb{R}^{m+r}$*, while* $\mathcal{U}'$ *is defined as a new uncertainty set as:*

$$\mathcal{U}' := \{\boldsymbol{\zeta}' \in \mathbb{R}^{n_\zeta + m} | P'\boldsymbol{\zeta}' \leq \boldsymbol{q}'\} \tag{21}$$

with

$$P' = \begin{bmatrix} P & 0 \\ -D & B^T \\ D & -B^T \end{bmatrix}, \qquad \textit{and} \qquad \boldsymbol{q}' := \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{d} \\ -\boldsymbol{d} \end{bmatrix}.$$

and where the matrices

$$C' := \begin{bmatrix} 0 & -A^T \end{bmatrix}, \qquad \boldsymbol{d}' := \begin{bmatrix} -\boldsymbol{\psi} \\ -\boldsymbol{v} \end{bmatrix}, \qquad \boldsymbol{f}' := \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\psi} \end{bmatrix},$$

$$A' := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad B' := \begin{bmatrix} A^T & W^T \\ -A^T & -W^T \\ B^T & 0 \\ -B^T & 0 \\ -I & 0 \\ 0 & -I \end{bmatrix}, \qquad \Psi' := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ D & 0 \\ -D & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \textit{and } \boldsymbol{\psi}' := \begin{bmatrix} \boldsymbol{c} \\ -\boldsymbol{c} \\ \boldsymbol{d} \\ -\boldsymbol{d} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

are considered. Furthermore, the TSLRO reformulation (20) *satisfies assumptions 1, 2, and 3 when the WCARM also satisfies assumptions 3 and 5, while the WCARM needs to additionally satisfy Assumption 4 for the TSLRO reformulation to satisfy Assumption 4.*

Proof. Let us consider the following maximization problem which is part of the WCARM problem with objective uncertainty:

$$\sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}',\boldsymbol{\zeta}) = \sup_{\boldsymbol{x}',\boldsymbol{y}'} \quad \boldsymbol{c}^T\boldsymbol{x}' + \boldsymbol{d}^T(\boldsymbol{\zeta})\boldsymbol{y}' \tag{22a}$$

$$\text{s.t.} \quad A\boldsymbol{x}' + B\boldsymbol{y}' \leq \boldsymbol{\psi} \tag{22b}$$

$$W\boldsymbol{x}' \leq \boldsymbol{v} \tag{22c}$$

Based on Assumption 2, there necessarily exists a $\boldsymbol{x}'$ and $\boldsymbol{y}'$ that make problem (22) feasible. Therefore, strong duality holds and the dual form of problem (22) can be derived by introducing the dual variables $\boldsymbol{\lambda} \in \mathbb{R}^m$ and $\boldsymbol{\gamma} \in \mathbb{R}^r$ associated with constraints (22b) and (22c), respectively. Thus, we obtain:

$$\sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}',\boldsymbol{\zeta}) = \inf_{\boldsymbol{\lambda}\geq 0,\boldsymbol{\gamma}\geq 0} \quad \boldsymbol{\psi}^T\boldsymbol{\lambda} + \boldsymbol{v}^T\boldsymbol{\gamma} \tag{23a}$$

$$\text{s.t.} \quad A^T\boldsymbol{\lambda} + W^T\boldsymbol{\gamma} = \boldsymbol{c} \tag{23b}$$

$$B^T\boldsymbol{\lambda} = \boldsymbol{d}(\boldsymbol{\zeta})\,. \tag{23c}$$

Since the strong duality property holds for both problems (11) and (22), it is possible to rewrite the WCARM problem by substituting both $h(\boldsymbol{x},\boldsymbol{\zeta})$ and $\sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}',\boldsymbol{\zeta})$ using there respective dual form, which results to the following reformulation:

$$\begin{aligned}
\text{WCARM} \;\equiv\; & \underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \quad \sup_{\boldsymbol{\zeta}\in\mathcal{U}} \left\{ \sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}',\boldsymbol{\zeta}) - h(\boldsymbol{x},\boldsymbol{\zeta}) \right\} \\
\equiv\; & \underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \quad \sup_{\boldsymbol{\zeta}\in\mathcal{U}} \left\{ \sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}',\boldsymbol{\zeta}) - \inf_{\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} \boldsymbol{c}^T\boldsymbol{x} + (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \right\} \\
\equiv\; & \underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \quad \sup_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} \left\{ \sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}',\boldsymbol{\zeta}) - \boldsymbol{c}^T\boldsymbol{x} + (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \right\} \\
\equiv\; & \underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \quad \sup_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} \; \inf_{(\boldsymbol{\lambda},\boldsymbol{\gamma})\in\Upsilon_1(\boldsymbol{\zeta})} \boldsymbol{\psi}^T\boldsymbol{\lambda} + \boldsymbol{v}^T\boldsymbol{\gamma} - \boldsymbol{c}^T\boldsymbol{x} - (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \\
\equiv\; & \underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \inf_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} \; \sup_{(\boldsymbol{\lambda},\boldsymbol{\gamma})\in\Upsilon_1(\boldsymbol{\zeta})} -\boldsymbol{\psi}^T\boldsymbol{\lambda} - \boldsymbol{v}^T\boldsymbol{\gamma} + \boldsymbol{c}^T\boldsymbol{x} + (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho}\,,
\end{aligned} \tag{24}$$

where $\Upsilon_1(\boldsymbol{\zeta}) := \{(\boldsymbol{\lambda},\boldsymbol{\gamma}) \in \mathbb{R}^m \times \mathbb{R}^r \,|\, \boldsymbol{\lambda} \geq 0, \boldsymbol{\gamma} \geq 0,\ (23\text{b}),\ (23\text{c})\}$ and $\Upsilon_2(\boldsymbol{\zeta}) := \{\boldsymbol{\rho} \in \mathbb{R}^m \,|\, B^T\boldsymbol{\rho} = \boldsymbol{d}(\boldsymbol{\zeta}),\ \boldsymbol{\rho} \geq 0\}$. By using the two liftings $\boldsymbol{\zeta}' = \begin{bmatrix}\boldsymbol{\zeta}\\ \boldsymbol{\rho}\end{bmatrix}$ and $\boldsymbol{y}'(\boldsymbol{\zeta}) := \begin{bmatrix}\boldsymbol{\lambda}(\boldsymbol{\zeta})\\ \boldsymbol{\gamma}(\boldsymbol{\zeta})\end{bmatrix}$, problem (24) can be rewritten in the form presented in equation (20).

Regarding the conditions on WCARM for the TSLRO reformulation to satisfy some of the stated assumptions, we start by considering that WCARM satisfies assumptions 1, 2, 3, and 5. Based on Assumption 3, it is possible to identify an $\bar{\boldsymbol{x}} \in \mathcal{X}$ and $\bar{\boldsymbol{\zeta}} \in \mathcal{U}$ such that $h(\bar{\boldsymbol{x}},\bar{\boldsymbol{\zeta}})$ is bounded. This implies by LP duality that there must be a feasible $\bar{\boldsymbol{\rho}} \in \Upsilon_2(\bar{\boldsymbol{\zeta}})$. Moreover, Assumption 5 implies that $\sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}',\bar{\boldsymbol{\zeta}})$ is bounded hence once again LP duality ensures that there exists a pair $(\bar{\boldsymbol{\lambda}},\bar{\boldsymbol{\gamma}}) \in \Upsilon_1(\bar{\boldsymbol{\zeta}})$. The TSLRO reformulation therefore satisfies Assumption 1 using the quintuplet $(\bar{\boldsymbol{x}},\bar{\boldsymbol{\zeta}},\bar{\boldsymbol{\rho}},\bar{\boldsymbol{\lambda}},\bar{\boldsymbol{\gamma}})$. Next, the fact that the TSLRO reformulation satisfies Assumption 2 follows similarly from imposing Assumption 5 on WCARM since the existence of a pair $(\bar{\boldsymbol{\lambda}},\bar{\boldsymbol{\gamma}}) \in \Upsilon_1(\boldsymbol{\zeta})$ holds for all $\boldsymbol{\zeta} \in \mathcal{U}$. Finally, Assumption 3 implies that there exists a $\bar{\boldsymbol{\zeta}}(\boldsymbol{x}) \in \mathcal{U}$ such that, for all $\boldsymbol{x} \in \mathcal{X}$, $h(\boldsymbol{x},\bar{\boldsymbol{\zeta}}(\boldsymbol{x})) < \infty$. From this we can conclude that:

$$\inf_{\boldsymbol{x}\in\mathcal{X}} \sup_{\boldsymbol{\zeta}\in\mathcal{U}} \left\{ \sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}',\boldsymbol{\zeta}) - h(\boldsymbol{x},\boldsymbol{\zeta}) \right\} \geq \inf_{\boldsymbol{x}\in\mathcal{X}} \sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}',\bar{\boldsymbol{\zeta}}(\boldsymbol{x})) - h(\boldsymbol{x},\bar{\boldsymbol{\zeta}}(\boldsymbol{x})) \geq 0 > -\infty\,.$$

The WCARM problem is therefore bounded below by zero hence the TSLRO reformulation is bounded above by zero which demonstrates that the latter satisfies Assumption 3.

Now, given that the WCARM additionally satisfies Assumption 4, we therefore have that for all $\boldsymbol{x} \in \mathcal{X}$:

$$\sup_{\boldsymbol{\zeta}\in\mathcal{U}} \left\{ \sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) - h(\boldsymbol{x}, \boldsymbol{\zeta}) \right\} \leq \left(\sup_{\boldsymbol{\zeta}\in\mathcal{U}} \sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) \right) - \left(\inf_{\boldsymbol{\zeta}\in\mathcal{U}} h(\boldsymbol{x}, \boldsymbol{\zeta}) \right) < \infty \,,$$

Where the first term is bounded above according to Assumption 5 and the second term bounded below according to Assumption 4. We can thus conclude that for all $\boldsymbol{x} \in \mathcal{X}$, the worst-case regret is bounded above, thus that for all $\boldsymbol{x} \in \mathcal{X}$ the "worst-case profit" achievable in the TSLRO reformulation is bounded below, i.e. Assumption 4 is satisfied by the TSLRO reformulation. □

Once again, Proposition 4 states that the WCARM model with objective uncertainty can be reformulated as a TSLRO problem and can therefore benefit from solution methods developed for adjustable robust optimization problems. In particular, a conservative approximation can be obtained using affine decision rules, which reduces to the linear program (8) when assumptions 1, 2, 3, and 5 are satisfied by the WCARM. In order to implement the column-and-constraint generation algorithm described in section 3.1.1, one needs to additionally verify that the WCARM satisfies Assumption 4.

5 TSLRO Reformulations for Worst-case Relative Regret Minimization Problems

An alternative form of regret minimization problem considers regret in its relative, rather than absolute, form. This approach is also equivalently measured according to a so-called "competitive ratio", which is a popular measure in the field of online optimization (Borodin and El-Yaniv, 2005). As defined in Kouvelis and Yu (1996), the worst-case relative regret criterion aims at evaluating the performance of a decision $\boldsymbol{x}$ with respect to the worst-case regret that might be experienced in hindsight relatively to the best decision that could have been made. Mathematically speaking, given a non-negative profit function $h(\boldsymbol{x}, \boldsymbol{\zeta})$ which depends on both the decision and the realization of some uncertain vector of parameters $\boldsymbol{\zeta}$, one measures the relative regret experienced once $\boldsymbol{\zeta}$ is revealed as the ratio of the difference between the best profit achievable $\sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})$ and the profit $h(\boldsymbol{x}, \boldsymbol{\zeta})$ achieved by the decision $\boldsymbol{x}$ that was implemented, over the best profit achievable. When Assumption 5 is satisfied, the worst-case relative regret minimization (WCRRM) problem thus takes the form:

$$\text{(WCRRM)} \quad \underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \; \sup_{\boldsymbol{\zeta}\in\mathcal{U}} \; \left\{ \frac{\sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) - h(\boldsymbol{x}, \boldsymbol{\zeta})}{\sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})} \right\} . \tag{25}$$

where it is understood that the relative regret is null if $\sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) = h(\boldsymbol{x}, \boldsymbol{\zeta}) = 0$. Mathematically speaking, we might be more accurate by defining the WCRRM problem as:

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \; \sup_{\boldsymbol{\zeta}\in\mathcal{U}} \; \lim_{\epsilon\to 0^+} \left\{ \frac{\sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) - h(\boldsymbol{x}, \boldsymbol{\zeta})}{\epsilon + \sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})} \right\} .$$

Besides Assumption 5, the following two assumptions will be useful in deriving TSLRO reformulations for WCRRM problems.

Assumption 6. *The profit function $h(\boldsymbol{x}, \boldsymbol{\zeta}) \geq 0$ for all $\boldsymbol{x} \in \mathcal{X}$ and all $\boldsymbol{\zeta} \in \mathcal{U}$. This implies that the WCRRM problem satisfies Assumption 2 and, with Assumption 5, that the optimal value of problem* (25) *lies in the closed interval* $[0, 1]$.

Assumption 7. *It is possible to achieve a strictly positive worst-case profit, namely*

$$\exists \boldsymbol{x} \in \mathcal{X}, \; \forall \boldsymbol{\zeta} \in \mathcal{U}, \; h(\boldsymbol{x}, \boldsymbol{\zeta}) > 0 \,.$$

Together with Assumption 5, this implies that the optimal value of problem (25) *lies in the open interval* $[0, 1[$.

While assumptions 5 and 6 simply formalize an hypothesis that needs to be made for problem WCRRM to be meaningful, we argue that Assumption 7 is made without loss of generality since if it is not the case, then the WCRRM becomes trivial. Indeed, one can then simply consider any $\boldsymbol{x} \in \mathcal{X}$ as an optimal solution to the WCRRM since it achieves the best possible worst-case relative regret, i.e. either 0% or 100%.

In what follows we demonstrate how the WCRRM problem can be reformulated as a TSLRO when the profit function $h(\boldsymbol{x}, \boldsymbol{\zeta})$ captures the profit of a second-stage linear decision model with either right-hand side or objective uncertainty. Note that for completeness Appendix B presents similar TSLRO reformulation for the case where the two-stage problem is a cost minimization problem, i.e. that $h(\boldsymbol{x}, \boldsymbol{\zeta})$ is non-positive.

5.1 The Case of Right-Hand Side Uncertainty

We consider the case where $h(\boldsymbol{x}, \boldsymbol{\zeta})$ takes the form presented in problem (3) and where uncertainty is limited to the right-hand side as defined in Definition 1.

Proposition 5. *Given that assumptions 1, 5, and 6 are satisfied, the WCRRM problem with right-hand side uncertainty is equivalent to the following TSLRO problem:*

$$\begin{aligned} \underset{\boldsymbol{x}' \in \mathcal{X}', \boldsymbol{y}'(\cdot)}{\text{maximize}} \quad & \inf_{\boldsymbol{\zeta}' \in \mathcal{U}'} \boldsymbol{c}'^T \boldsymbol{x}' && \text{(26a)} \\ \text{subject to} \quad & A' \boldsymbol{x}' + B' \boldsymbol{y}'(\boldsymbol{\zeta}') \leq \Psi'(\boldsymbol{x}')\boldsymbol{\zeta}' + \boldsymbol{\psi}', \, \forall \, \boldsymbol{\zeta}' \in \mathcal{U}' \,, && \text{(26b)} \end{aligned}$$

where $x' \in \mathbb{R}^{n_x+1}$*,* $\boldsymbol{\zeta}' \in \mathbb{R}^{n_\zeta + n_x + n_y}$*,* $\boldsymbol{y}' : \mathbb{R}^{n_\zeta + n_x + n_y} \to \mathbb{R}^{n_y}$*,* $\boldsymbol{c}' = [-1 \quad \boldsymbol{0}^T]^T$*, while* $\mathcal{X}' := \{[t \;\; \boldsymbol{x}^T]^T \in \mathbb{R}^{n_x+1} \,|\, \boldsymbol{x} \in \mathcal{X}, \, t \in [0,\, 1]\}$*,* $\mathcal{U}'$ *is defined as in equation* (17) *and*

$$A' = \begin{bmatrix} 0 & -\boldsymbol{c}^T \\ 0 & A \end{bmatrix}, \quad B' = \begin{bmatrix} -\boldsymbol{d}^T \\ B \end{bmatrix}, \quad \Psi'(\boldsymbol{x}') = \begin{bmatrix} \boldsymbol{0}^T & -\boldsymbol{c}^T & -\boldsymbol{d}^T \\ \Psi & 0 & 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{0}^T & \boldsymbol{c}^T & \boldsymbol{d}^T \\ 0 & 0 & 0 \end{bmatrix} x_1', \quad \boldsymbol{\psi}' := \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\psi} \end{bmatrix}.$$

In particular, a solution for the WCRRM takes the form of $\boldsymbol{x}^* := \boldsymbol{x}'^*_{2:n_x+1}$ *and achieves a worst-case relative regret of* x_1'*. Furthermore, this TSLRO reformulation necessarily satisfies Assumption 1 while it only satisfies Assumption 2 if all* $\boldsymbol{x} \in \mathcal{X}$ *achieve a worst-case regret of zero.*

Proof. We first employ an epigraph form for problem (25) as follows:

$$\begin{aligned} \underset{\boldsymbol{x} \in \mathcal{X}, t}{\text{minimize}} \quad & t && \text{(27a)} \\ \text{subject to} \quad & \sup_{\boldsymbol{\zeta} \in \mathcal{U}} \left\{ \frac{\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) - h(\boldsymbol{x}, \boldsymbol{\zeta})}{\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})} \right\} \leq t && \text{(27b)} \\ & 0 \leq t \leq 1 \,, && \text{(27c)} \end{aligned}$$

where we impose that $t \in [0,\, 1]$ since assumptions 5 and 6 ensures that the optimal value of the WCRRM problem is in $[0,\, 1]$. One can then manipulate constraint (27b) to show that it is equivalent to

$$\frac{\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) - h(\boldsymbol{x}, \boldsymbol{\zeta})}{\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})} \leq t \,, \, \forall \boldsymbol{\zeta} \in \mathcal{U} \,,$$

and moreover to

$$\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) - h(\boldsymbol{x}, \boldsymbol{\zeta}) \leq t(\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})) \,, \, \forall \boldsymbol{\zeta} \in \mathcal{U} \,,$$

since it is clearly the case if $\boldsymbol{\zeta}$ is such that $\sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}',\boldsymbol{\zeta}) > 0$ and otherwise would lead to the constraint that $-h(\boldsymbol{x},\boldsymbol{\zeta}) \leq 0$ which is necessarily satisfied and is coherent with the fact that we consider regret to be equal to 0 for such a $\boldsymbol{\zeta}$. Finally, we obtain the constraint:

$$(1-t)\sup_{\boldsymbol{x}'\in\mathcal{X}} h(\boldsymbol{x}',\boldsymbol{\zeta}) - h(\boldsymbol{x},\boldsymbol{\zeta}) \leq 0\ ,\ \forall \boldsymbol{\zeta}\in\mathcal{U}\,. \tag{28}$$

By substituting problem (3) in this constraint we obtain the following reformulations

$$\begin{aligned}
(27\text{b}) \ \equiv\ & (1-t)\sup_{\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta})} \boldsymbol{c}^T\boldsymbol{x}' + \boldsymbol{d}^T\boldsymbol{y}' - \sup_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} \leq 0\ ,\ \forall\boldsymbol{\zeta}\in\mathcal{U}\\
\equiv\ & \inf_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} (1-t)\boldsymbol{c}^T\boldsymbol{x}' + (1-t)\boldsymbol{d}^T\boldsymbol{y}' - \boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{d}^T\boldsymbol{y} \leq 0\ ,\ \forall\boldsymbol{\zeta}\in\mathcal{U}, \boldsymbol{x}'\in\mathcal{X}, \boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta})\,.
\end{aligned}$$

Hence the WCRRM problem reduces to:

$$\underset{\boldsymbol{x}\in\mathcal{X},t\in[0,\,1]}{\text{minimize}} \quad \sup_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta})} h'(\boldsymbol{x},t,\boldsymbol{\zeta},\boldsymbol{x}',\boldsymbol{y}')\,,$$

where

$$\begin{aligned}
h'(\boldsymbol{x},t,\boldsymbol{\zeta},\boldsymbol{x}',\boldsymbol{y}') := \quad & \inf_{\boldsymbol{y}} \quad t\\
& \text{s.t.} \quad -\boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{d}^T\boldsymbol{y} \leq -(1-t)\boldsymbol{c}^T\boldsymbol{x}' - (1-t)\boldsymbol{d}^T\boldsymbol{y}'\\
& \qquad\ \ A\boldsymbol{x} + B\boldsymbol{y} \leq \Psi\boldsymbol{\zeta} + \boldsymbol{\psi}\,.
\end{aligned}$$

This problem can be rewritten in the form presented in equation (26).

Regarding the assumptions that are satisfied by this TSLRO reformulation, we can straightforwardly verify that based on Assumption 1, there must be a triplet $(\bar{\boldsymbol{x}},\bar{\boldsymbol{\zeta}},\bar{\boldsymbol{y}})$ such that $\bar{\boldsymbol{x}}\in\mathcal{X}$, $\bar{\boldsymbol{\zeta}}\in\mathcal{U}$, and $\bar{\boldsymbol{y}}\in\mathcal{Y}(\bar{\boldsymbol{x}},\bar{\boldsymbol{\zeta}})$ and construct an assignment for $\bar{\boldsymbol{x}}' := \bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}' := \bar{\boldsymbol{y}}$ and $\bar{t} := 0$ which satisfy all the constraints of the new TSLRO reformulation. Unfortunately, if there exists an $\boldsymbol{x}\in\mathcal{X}$ such that the worst-case relative regret is strictly greater than 0, then there clearly exists a $\bar{\tau} > 0$ and a feasible triplet $(\bar{\boldsymbol{\zeta}},\bar{\boldsymbol{x}}',\bar{\boldsymbol{y}}')$ for which the recourse problem $h'(\boldsymbol{x},\bar{\tau},\bar{\boldsymbol{\zeta}},\bar{\boldsymbol{x}}',\bar{\boldsymbol{y}}')$ becomes infeasible, hence the new TSLRO reformulation does not satisfy Assumption 2. □

Proposition 5 motivates the application of solution methods developed for adjustable robust optimization problems to WCRRM problems. It is clear for instance that a conservative approximation that takes the form of the linear program (8) can readily be obtained by using affine decision rules. Exact methods however must be designed in a way that can handle TSLRO problems that do not satisfy relatively complete recourse. In particular, in our numerical experiments we will make use of the method proposed in Ayoub and Poss (2016).

5.2 The Case of Objective Uncertainty

We consider the case where $h(\boldsymbol{x},\boldsymbol{\zeta})$ takes the form presented in problem (11).

Proposition 6. *Given that assumptions 1, 5, 6, and 7 are satisfied, the WCRRM problem with objective uncertainty is equivalent to the following TSLRO problem:*

$$\begin{aligned}
\underset{\boldsymbol{x}',\boldsymbol{y}'(\cdot)}{\text{maximize}} \quad & \inf_{\boldsymbol{\zeta}'\in\mathcal{U}'} \boldsymbol{c}'^T\boldsymbol{x}' && \text{(29a)}\\
\text{subject to} \quad & A'\boldsymbol{x} + B'\boldsymbol{y}'(\boldsymbol{\zeta}') \leq \Psi'(\boldsymbol{x}')\boldsymbol{\zeta}' + \boldsymbol{\psi}' && \text{(29b)}\\
& \boldsymbol{x}'\in\mathcal{X}', && \text{(29c)}
\end{aligned}$$

where $\boldsymbol{x}' \in \mathbb{R}^{n_x+1}$, $\boldsymbol{y}' : \mathbb{R}^{n_\zeta+m} \to \mathbb{R}^{m+r}$, *while* $\mathcal{X}' := \{[u\ \ \boldsymbol{z}^T]^T \in \mathbb{R}^{n_x+1} \,|\, W\boldsymbol{z} \leq \boldsymbol{v}u,\, u \geq 1\}$, $\mathcal{U}'$ *is defined as in equation (21). Furthermore, we have that* $\boldsymbol{c}' := [-1\ \ \boldsymbol{0}^T]^T$, *while*

$$A' := \begin{bmatrix} 0 & -\boldsymbol{c}^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\quad B' := \begin{bmatrix} \boldsymbol{\psi}^T & \boldsymbol{v}^T \\ A^T & W^T \\ -A^T & -W^T \\ B^T & 0 \\ -B^T & 0 \\ -I & 0 \\ 0 & -I \end{bmatrix},\quad \Psi'(\boldsymbol{x}') := \begin{bmatrix} \boldsymbol{0}^T & \boldsymbol{\psi}^T x_1' - \boldsymbol{x}'^T_{2:n_x+1} A^T \\ 0 & 0 \\ 0 & 0 \\ D & 0 \\ -D & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\quad \text{and } \boldsymbol{\psi}' := \begin{bmatrix} 0 \\ \boldsymbol{c} \\ -\boldsymbol{c} \\ \boldsymbol{d} \\ -\boldsymbol{d} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}.$$

are considered. In particular, a solution for the WCRRM takes the form of $\boldsymbol{x}^* := \boldsymbol{x}'^*_{2:n_x+1}/x_1'^*$ *and achieves a worst-case relative regret of* $1 - 1/x_1'^*$. *Finally, this TSLRO reformulation necessarily satisfies Assumption 1 while it only satisfies Assumption 2 if all* $\boldsymbol{x} \in \mathcal{X}$ *achieve worst-case regret of zero.*

Proof. The first steps of this proof are exactly as in the proof of Proposition 5 up to equation (28), except for the small difference that we will consider $t \in [0, 1[$ which follows from Assumption 7. Since we are now dealing with objective uncertainty, we substitute $h(\boldsymbol{x}, \boldsymbol{\zeta})$ and $\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})$ using there respective dual form (see equations (13) and (23) respectively), where strong duality follows again from Assumption 2 implied by Assumption 6. This leads to the following reformulation:

$$\begin{aligned}
\text{(27b)} \ &\equiv\ (1-t) \sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) - h(\boldsymbol{x}, \boldsymbol{\zeta}) \leq 0\ ,\ \forall \boldsymbol{\zeta} \in \mathcal{U} & (30)\\
&\equiv\ (1-t)\left(\inf_{(\boldsymbol{\lambda},\boldsymbol{\gamma}) \in \Upsilon_1(\boldsymbol{\zeta})} \boldsymbol{\psi}^T\boldsymbol{\lambda} + \boldsymbol{v}^T\boldsymbol{\gamma}\right) - \inf_{\boldsymbol{\rho} \in \Upsilon_2(\boldsymbol{\zeta})} \{\boldsymbol{c}^T\boldsymbol{x} + (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho}\} \leq 0, \forall \boldsymbol{\zeta} \in \mathcal{U} & (31)\\
&\equiv \inf_{(\boldsymbol{\lambda},\boldsymbol{\gamma}) \in \Upsilon_1(\boldsymbol{\zeta})} (1-t)(\boldsymbol{\psi}^T\boldsymbol{\lambda} + \boldsymbol{v}^T\boldsymbol{\gamma}) - \boldsymbol{c}^T\boldsymbol{x} - (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \leq 0\ ,\ \forall \boldsymbol{\zeta} \in \mathcal{U}, \forall \boldsymbol{\rho} \in \Upsilon_2(\boldsymbol{\zeta}) & (32)\\
&\equiv \inf_{(\boldsymbol{\lambda},\boldsymbol{\gamma}) \in \Upsilon_1(\boldsymbol{\zeta})} \boldsymbol{\psi}^T\boldsymbol{\lambda} + \boldsymbol{v}^T\boldsymbol{\gamma} - \frac{1}{1-t}\boldsymbol{c}^T\boldsymbol{x} - \frac{1}{1-t}(\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \leq 0, \forall \boldsymbol{\zeta} \in \mathcal{U}, \forall \boldsymbol{\rho} \in \Upsilon_2(\boldsymbol{\zeta})\,, & (33)
\end{aligned}$$

where $\Upsilon_1(\boldsymbol{\zeta})$ and $\Upsilon_2(\boldsymbol{\zeta})$ are as defined in the proof of Proposition 4. Hence the WCRRM problem reduces to:

$$\underset{\boldsymbol{x} \in \mathcal{X}, t \in [0,\,1[}{\text{minimize}} \quad \sup_{\boldsymbol{\zeta} \in \mathcal{U}, \boldsymbol{\rho} \in \Upsilon_2(\boldsymbol{\zeta})} h'(\boldsymbol{x}, t, \boldsymbol{\zeta}, \boldsymbol{\rho})\,,$$

where

$$\begin{aligned}
h'(\boldsymbol{x}, t, \boldsymbol{\zeta}, \boldsymbol{\rho}) := \quad & \inf_{\boldsymbol{\lambda},\boldsymbol{\gamma}} \quad t \\
& \text{s.t.} \quad \boldsymbol{\psi}^T\boldsymbol{\lambda} + \boldsymbol{v}^T\boldsymbol{\gamma} - \frac{1}{1-t}\boldsymbol{c}^T\boldsymbol{x} - \frac{1}{1-t}(\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \leq 0 \\
& \qquad A^T\boldsymbol{\lambda} + W^T\boldsymbol{\gamma} = \boldsymbol{c} \\
& \qquad B^T\boldsymbol{\lambda} = \boldsymbol{d}(\boldsymbol{\zeta}) \\
& \qquad \boldsymbol{\lambda} \geq 0,\ \boldsymbol{\gamma} \geq 0\,.
\end{aligned}$$

Using a simple replacement of variables $u := 1/(1-t)$ and $\boldsymbol{z} := (1/(1-t))\boldsymbol{x}$ and applying a monotone transformation of the objective function $t \to 1/(1-t)$, we obtain that the WCRRM is equivalently represented as

$$\underset{u \geq 1,\, \boldsymbol{z}: W\boldsymbol{z} \leq \boldsymbol{v}u}{\text{minimize}} \quad \sup_{\boldsymbol{\zeta} \in \mathcal{U}, \boldsymbol{\rho} \in \Upsilon_2(\boldsymbol{\zeta})} h''(\boldsymbol{z}, u, \boldsymbol{\zeta}, \boldsymbol{\rho})$$

where

$$
\begin{aligned}
h''(\boldsymbol{z}, u, \boldsymbol{\zeta}, \boldsymbol{\rho}) := \quad & \inf_{\boldsymbol{\lambda}, \boldsymbol{\gamma}} && u && \text{(34a)} \\
& \text{s.t.} && \boldsymbol{\psi}^T \boldsymbol{\lambda} + \boldsymbol{v}^T \boldsymbol{\gamma} - \boldsymbol{c}^T \boldsymbol{z} - (\boldsymbol{\psi} u - A\boldsymbol{z})^T \boldsymbol{\rho} \leq 0 && \text{(34b)} \\
& && A^T \boldsymbol{\lambda} + W^T \boldsymbol{\gamma} = \boldsymbol{c} && \text{(34c)} \\
& && B^T \boldsymbol{\lambda} = \boldsymbol{d}(\boldsymbol{\zeta}) && \text{(34d)} \\
& && \boldsymbol{\lambda} \geq 0,\ \boldsymbol{\gamma} \geq 0\,. && \text{(34e)}
\end{aligned}
$$

This problem can be rewritten in the form presented in equation (29).

Regarding the assumptions that are satisfied by this TSLRO reformulation, we can straightforwardly verify that based on Assumption 1, there must be a triplet $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{y}})$ such that $\bar{\boldsymbol{x}} \in \mathcal{X}$, $\bar{\boldsymbol{\zeta}} \in \mathcal{U}$, and $\bar{\boldsymbol{y}} \in \mathcal{Y}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\zeta}})$ and construct an assignment for $\bar{\boldsymbol{x}}' := \bar{\boldsymbol{x}}$, $\bar{\boldsymbol{y}}' := \bar{\boldsymbol{y}}$, $\bar{\boldsymbol{z}} := \bar{\boldsymbol{x}}$, and $\bar{u} := 1$ which satisfy all the constraints of the TSLRO reformulation. Finally, the difficulties of satisfying Assumption 2 can be demonstrated exactly as in the proof of Proposition 5. □

This final proposition reformulating WCRRM problems with objective uncertainty as TSLRO problems motivates once more the application of solution methods developed for adjustable robust optimization problems to this underexplored class of problems. In particular, a tractable conservative approximation can directly be obtained by using affine decision rules while to obtain an exact solution, a method as proposed in Ayoub and Poss (2016) needs to be employed.

6 Optimality of Affine Decision Rules

In this section, we derive conditions under which one can establish that affine decision rules are optimal in the TSLRO reformulation of WCARM and WCRRM problems. These results will draw there arguments from similar results that have been established for two-stage robust optimization. In fact, perhaps the most famous of those result is attributed to Bertsimas and Goyal (2012) for the case where the uncertainty set takes the form of a simplex set.

Definition 2. *An uncertainty set $\mathcal{U}$ is called a "simplex set" if it is the convex hull of $n_\zeta + 1$ affinely independent points in $\mathbb{R}^{n_\zeta}$.*

One can in fact extend the known optimality of affine decision to a special class of WCARM problems.

Proposition 7. *If $h(\boldsymbol{x}, \boldsymbol{\zeta})$ satisfies $\max_{\boldsymbol{x} \in \mathcal{X}} h(\boldsymbol{x}, \boldsymbol{\zeta}) = \boldsymbol{\gamma}^T \boldsymbol{\zeta} + \bar{\gamma}$ for some $\boldsymbol{\gamma} \in \mathbb{R}^{n_\zeta}$ and $\bar{\gamma} \in \mathbb{R}$ and $\mathcal{U}$ is a simplex set, then affine decision rules are optimal in the TSLRO reformulation of the WCARM (under Assumption 1) and WCRRM (under assumptions 1, 5, and 6) problems with right-hand side uncertainty, i.e. problem* (16) *and* (26) *respectively.*

Note that the condition that $\max_{\boldsymbol{x} \in \mathcal{X}} h(\boldsymbol{x}, \boldsymbol{\zeta}) = \boldsymbol{\gamma}^T \boldsymbol{\zeta} + \bar{\gamma}$ is satisfied in a number of classical inventory models. For instance, one has that this condition is satisfied for the following multi-item newsvendor problem (see Ardestani-Jaafari and Delage 2016):

$$
\underset{\boldsymbol{x} \geq 0}{\text{maximize}} \quad \inf_{\boldsymbol{\zeta} \in \mathcal{U}} \sum_{i=1}^{n_y} (p_i - c_i) x_i + \min(-b_i(\zeta_i - x_i), (s_i - p_i)(x_i - \zeta_i))\,,
$$

and where x_i is the number of units of item i ordered, ζ_i is the unknown demand for item i, p_i is the sales price for item i, c_i is the ordering cost, b_i is the shortage cost, and s_i is the salvage cost. Exploiting a well-known epigraph formulation, the single-stage model can be reformulated using $\max_{\boldsymbol{x}\geq 0}\min_{\boldsymbol{\zeta}\in\mathcal{U}} h(\boldsymbol{x},\boldsymbol{\zeta})$ with

$$\begin{aligned} h(\boldsymbol{x},\boldsymbol{\zeta}) := \max_{\boldsymbol{y}} \quad & \sum_{i}^{n_y} y_i \\ \text{subject to} \quad & y_i \leq (p_i - c_i)x_i + (s_i - p_i)(x_i - \zeta_i),\, \forall i = 1,\ldots,n_y \\ & y_i \leq (p_i - c_i)x_i - b_i(\zeta_i - x_i),\, \forall i = 1,\ldots,n_y\,. \end{aligned}$$

It is usually assumed that $s_i \leq c_i \leq p_i$, namely that the salvage price is smaller than the ordering cost which is itself smaller than retail price, so that if the demand vector $\boldsymbol{\zeta}$ was known then the optimal order would simply be $x_i^* = \zeta_i \mathbf{1}\{p_i - c_i + b_i \geq 0\}$. Hence, we have that:

$$\max_{\boldsymbol{x}\geq 0} h(\boldsymbol{x},\boldsymbol{\zeta}) = \sum_{i=1}^{n_y} \left(-b_i + (p_i - c_i + b_i)\mathbf{1}\{p_i - c_i + b_i \geq 0\}\right)\zeta_i\,.$$

Similarly, in a classical lot-sizing problem with backlog described as:

$$\underset{\boldsymbol{x}\geq 0}{\text{maximize}} \;\; \inf_{\boldsymbol{\zeta}\in\mathcal{U}} \sum_{t=1}^{T} -c_t x_t - \min\left(h_t(\sum_{t'=1}^{t} x_{t'} - \zeta_{t'}), b_t(\sum_{t'=1}^{t} \zeta_{t'} - x_{t'})\right)\,,$$

where x_t is the number of units ordered for time t, ζ_t is the demand for time t, while c_t is the ordering cost, h_t the holding cost, and b_t the shortage cost. One can exploit the well-known facility location reformulation (see for instance Pochet and Wolsey 1988) to simplify the full information problem:

$$\begin{aligned} \max_{\boldsymbol{x}\in\mathcal{X}} h(\boldsymbol{x},\boldsymbol{\zeta}) &= \max_{X: X\geq 0, \sum_{t=1}^{T+1} X_{t,t'} = \zeta'_t,\,\forall t'} \; -\left(\sum_{t=1}^{T} c_t \sum_{t'=1}^{T} X_{t,t'} + \sum_{i=1}^{t}\sum_{j=t+1}^{T} h_t X_{i,j} + \sum_{i=1}^{t}\sum_{j=t+1}^{T+1} b_t X_{j,i}\right) \\ &= -\left(\sum_{t=1}^{T} \min_{\boldsymbol{x}:\boldsymbol{x}\geq 0, \sum_{t'=1}^{T+1} x_{t'} = \zeta_t} \; \sum_{t'=1}^{T} c_{t'} x_{t'} + \sum_{t'=1}^{t-1}\sum_{t''=t'}^{t-1} h_{t''} x_{t'} + \sum_{t'=t+1}^{T+1}\sum_{t''=t}^{t'-1} b_{t''} x_{t'}\right) \\ &= -\sum_{t=1}^{T} \zeta_t \left(\min_{\boldsymbol{x}:\boldsymbol{x}\geq 0, \sum_{t'=1}^{T+1} x_{t'} = 1} \; \sum_{t'=1}^{T} c_{t'} x_{t'} + \sum_{t'=1}^{t-1}\sum_{t''=t'}^{t-1} h_{t''} x_{t'} + \sum_{t'=t+1}^{T+1}\sum_{t''=t}^{t'-1} b_{t''} x_{t'}\right)\,, \end{aligned}$$

where $X_{t,t'}$ captures the number of units produced at time t to satisfy the demand at time t'. We see once again that the optimal value is linear with respect to $\boldsymbol{\zeta}$.

Proof. Starting with the case of the WCARM problem, we let $h_1(\boldsymbol{x})$ be defined as the worst-case absolute regret achieved by $\boldsymbol{x}$ which can be captured in the following form based on Proposition 2:

$$h_1(\boldsymbol{x}) := \inf_{\boldsymbol{\zeta}'\in\mathcal{U}'} \sup_{\boldsymbol{y}\in\mathcal{Y}'(\boldsymbol{x},\boldsymbol{\zeta}')} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} + \boldsymbol{f}'^T\boldsymbol{\zeta}'\,,$$

where

$$\mathcal{Y}'(\boldsymbol{x},\boldsymbol{\zeta}') := \{\boldsymbol{y}\,|\,A\boldsymbol{x} + B\boldsymbol{y} \leq \Psi'\boldsymbol{\zeta}' + \boldsymbol{\psi}\}\,.$$

Alternatively, let $h_2(\boldsymbol{x})$ denote the conservative approximation of $h_1(\boldsymbol{x})$ obtained using affine decision rules:

$$h_2(\boldsymbol{x}) := \sup_{(\boldsymbol{y},Y_{\zeta'})\in\mathcal{Y}'_{\text{aff}}(\boldsymbol{x})} \inf_{\boldsymbol{\zeta}'\in\mathcal{U}'} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T(\boldsymbol{y} + Y_{\zeta'}\boldsymbol{\zeta}') + \boldsymbol{f}'^T\boldsymbol{\zeta}'\,,$$

with

$$\mathcal{Y}'_{\text{aff}}(\boldsymbol{x}) := \{(\boldsymbol{y},Y_{\zeta'})\,|\,A\boldsymbol{x} + B(\boldsymbol{y} + Y_{\zeta'}\boldsymbol{\zeta}') \leq \Psi'\boldsymbol{\zeta}' + \boldsymbol{\psi},\, \forall\,\boldsymbol{\zeta}'\in\mathcal{U}'\}\,.$$

Necessarily, we have that $h_1(\boldsymbol{x}) \geq h_2(\boldsymbol{x})$ since affine decision rules provide a conservative approximation. In order to demonstrate that $h_1(\boldsymbol{x}) = h_2(\boldsymbol{x})$, we are left with showing that $h_2(\boldsymbol{x}) \geq h_1(\boldsymbol{x})$ and proceed as follows:

$$\begin{aligned}
h_2(\boldsymbol{x}) \geq & \sup_{(\boldsymbol{y},[Y_\zeta\ 0\ 0])\in\mathcal{Y}'_{\text{aff}}(\boldsymbol{x})} \min_{\boldsymbol{\zeta}'\in\mathcal{U}'} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T(\boldsymbol{y} + [Y_\zeta\ 0\ 0]\boldsymbol{\zeta}') + \boldsymbol{f}'^T\boldsymbol{\zeta}' \\
= & \sup_{(\boldsymbol{y},[Y_\zeta\ 0\ 0])\in\mathcal{Y}'_{\text{aff}}(\boldsymbol{x})} \min_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}',\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T(\boldsymbol{y} + Y_\zeta\boldsymbol{\zeta}) - \boldsymbol{c}^T\boldsymbol{x}' - \boldsymbol{d}^T\boldsymbol{y}' \\
= & \sup_{(\boldsymbol{y},[Y_\zeta\ 0\ 0])\in\mathcal{Y}'_{\text{aff}}(\boldsymbol{x})} \min_{\boldsymbol{\zeta}\in\mathcal{U}} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T(\boldsymbol{y} + Y_\zeta\boldsymbol{\zeta}) - \max_{\boldsymbol{x}',\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} \boldsymbol{c}^T\boldsymbol{x}' + \boldsymbol{d}^T\boldsymbol{y}' \\
= & \sup_{(\boldsymbol{y},Y_\zeta)\in\mathcal{Y}_{\text{aff}}(\boldsymbol{x})} \min_{\boldsymbol{\zeta}\in\mathcal{U}} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T(\boldsymbol{y} + Y_\zeta\boldsymbol{\zeta}) - \boldsymbol{\gamma}^T\boldsymbol{\zeta} - \bar{\gamma} \qquad (35) \\
= & \sup_{t,(\boldsymbol{y},Y_\zeta)\in\mathcal{Y}_{\text{aff}}(\boldsymbol{x})} t \qquad (36) \\
& \quad \text{s.t.} \quad t \leq \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T(\boldsymbol{y} + Y_\zeta\boldsymbol{\zeta}) - \boldsymbol{\gamma}^T\boldsymbol{\zeta} - \bar{\gamma}\,,\ \forall\boldsymbol{\zeta}\in\mathcal{U} \\
= & \max_{t,\boldsymbol{y}(\cdot)} \ t \qquad (37) \\
& \text{s.t.} \quad t \leq \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y}(\boldsymbol{\zeta}) - \boldsymbol{\gamma}^T\boldsymbol{\zeta} - \bar{\gamma}\,,\ \forall\boldsymbol{\zeta}\in\mathcal{U} \\
& \qquad\ \ \boldsymbol{y}(\boldsymbol{\zeta}) \in \mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})\,,\ \forall\,\boldsymbol{\zeta}\in\mathcal{U} \\
= & \min_{\boldsymbol{\zeta}\in\mathcal{U}} \max_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} - \boldsymbol{\gamma}^T\boldsymbol{\zeta} - \bar{\gamma} \qquad (38) \\
= & \min_{\boldsymbol{\zeta}\in\mathcal{U}} \max_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} - \max_{x\in\mathcal{X}} h(\boldsymbol{x},\boldsymbol{\zeta}) \\
= & \ h_1(\boldsymbol{x})\,,
\end{aligned}$$

where

$$\mathcal{Y}_{\text{aff}} := \{(\boldsymbol{y}, Y_\zeta) \,|\, A\boldsymbol{x} + B(\boldsymbol{y} + Y_\zeta) \leq \Psi\boldsymbol{\zeta} + \boldsymbol{\psi},\ \forall\,\boldsymbol{\zeta}\in\mathcal{U}\}\,.$$

Detailing each step, we first obtained a lower bound by maximizing over a subset of the available affine decision rules. We then in the next three steps exploited the property that $\max_{x\in\mathcal{X}} h(\boldsymbol{x},\boldsymbol{\zeta}) = \boldsymbol{\gamma}^T\boldsymbol{\zeta} + \bar{\gamma}$. The fourth step consists in using an epigraph representation to cast the model in the form where all the uncertainty appears in the right-hand side. The equivalence between (36) and (37) follows from the fact that affine decision rules are optimal in two-stage robust linear programs with right-hand side uncertainty when the uncertainty set is a simplex set (see Theorem 1 in Bertsimas and Goyal 2012). Finally the steps are completed by replacing back $\boldsymbol{\gamma}^T\boldsymbol{\zeta} + \bar{\gamma} = \max_{x\in\mathcal{X}} h(\boldsymbol{x},\boldsymbol{\zeta})$ to obtain the expression of worst-case absolute regret which was defined as $h_1(\boldsymbol{x})$.

In the case of WCRRM, we can follow a similar reasoning. For any fixed $\boldsymbol{x}$ and t, we can let

$$\begin{aligned}
h_1(\boldsymbol{x},t) := \sup_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta})} \inf_{\boldsymbol{y}} \quad & t \\
\text{s.t.} \quad & -\boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{d}^T\boldsymbol{y} \leq -(1-t)\boldsymbol{c}^T\boldsymbol{x}' - (1-t)\boldsymbol{d}^T\boldsymbol{y}' \\
& A\boldsymbol{x} + B\boldsymbol{y} \leq \Psi\boldsymbol{\zeta} + \boldsymbol{\psi}\,.
\end{aligned}$$

and $h_2(\boldsymbol{x},t)$ as the upper bound obtained when applying affine decision rules of the form

$\boldsymbol{y}(\boldsymbol{\zeta},\boldsymbol{x}',\boldsymbol{y}') := \boldsymbol{y} + Y_\zeta\boldsymbol{\zeta} + Y_{x'}\boldsymbol{x}' + Y_{y'}\boldsymbol{y}'$. In this context we can show that

$$
\begin{aligned}
h_2(\boldsymbol{x},t) \leq \inf_{\boldsymbol{y},Y_\zeta} \;& t \\
\text{s.t.} \;& -\boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{d}^T(\boldsymbol{y}+Y_\zeta\boldsymbol{\zeta}) \leq -(1-t)\boldsymbol{c}^T\boldsymbol{x}' - (1-t)\boldsymbol{d}^T\boldsymbol{y}'\,,\,\forall\,\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta}) \\
& A\boldsymbol{x} + B(\boldsymbol{y}+Y_\zeta\boldsymbol{\zeta}) \leq \Psi\boldsymbol{\zeta} + \boldsymbol{\psi}\,,\,\forall\,\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta}) \\
= \inf_{\boldsymbol{y},Y_\zeta} \;& t \\
\text{s.t.} \;& -\boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{d}^T(\boldsymbol{y}+Y_\zeta\boldsymbol{\zeta}) \leq -(1-t)(\boldsymbol{c}^T\boldsymbol{x}' + \boldsymbol{d}^T\boldsymbol{y}')\,,\,\forall\,\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta}) \\
& A\boldsymbol{x} + B(\boldsymbol{y}+Y_\zeta\boldsymbol{\zeta}) \leq \Psi\boldsymbol{\zeta} + \boldsymbol{\psi}\,,\,\forall\,\boldsymbol{\zeta}\in\mathcal{U} \\
= \inf_{\boldsymbol{y},Y_\zeta} \;& t \\
\text{s.t.} \;& -\boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{d}^T(\boldsymbol{y}+Y_\zeta\boldsymbol{\zeta}) \leq -(1-t)(\bar{\gamma} + \boldsymbol{\gamma}^T\boldsymbol{\zeta})\,,\,\forall\,\boldsymbol{\zeta}\in\mathcal{U} \\
& A\boldsymbol{x} + B(\boldsymbol{y}+Y_\zeta\boldsymbol{\zeta}) \leq \Psi\boldsymbol{\zeta} + \boldsymbol{\psi}\,,\,\forall\,\boldsymbol{\zeta}\in\mathcal{U} \\
= \;& \begin{cases} t & \text{if } \sup_{(\boldsymbol{y},Y_\zeta)\in\mathcal{Y}_{\text{aff}}(\boldsymbol{x})} \; \min_{\boldsymbol{\zeta}\in\mathcal{U}} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T(\boldsymbol{y}+Y_\zeta\boldsymbol{\zeta}) - (1-t)(\boldsymbol{\gamma}^T\boldsymbol{\zeta} - \bar{\gamma}) \geq 0 \\ \infty & \text{otherwise} \end{cases} \\
= \;& \begin{cases} t & \text{if } \min_{\boldsymbol{\zeta}\in\mathcal{U}} \max_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} \; \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} - (1-t)(\boldsymbol{\gamma}^T\boldsymbol{\zeta} - \bar{\gamma}) \geq 0 \\ \infty & \text{otherwise} \end{cases} \qquad (39) \\
= \;& \begin{cases} t & \text{if } \min_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta})} \max_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x},\boldsymbol{\zeta})} \; \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T(\boldsymbol{y}+Y_\zeta\boldsymbol{\zeta}) - (1-t)(\boldsymbol{c}^T\boldsymbol{x}' + \boldsymbol{d}^T\boldsymbol{y}') \geq 0 \\ \infty & \text{otherwise} \end{cases} \\
& \qquad (40) \\
= \;& \sup_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}',\boldsymbol{\zeta})} \; \inf_{\boldsymbol{y}} \; t \\
& \qquad \text{s.t.} \; -\boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{d}^T\boldsymbol{y} \leq -(1-t)(\boldsymbol{c}^T\boldsymbol{x}' + \boldsymbol{d}^T\boldsymbol{y}') \\
& \qquad\qquad A\boldsymbol{x} + B\boldsymbol{y} \leq \Psi\boldsymbol{\zeta} + \boldsymbol{\psi} \\
= \;& h_1(\boldsymbol{x},t)\,,
\end{aligned}
$$

where the equivalence between (39) and (40) was already demonstrated in going through equations (35) to (38).

□

Proposition 7 also has an analog in the context of a two-stage model with objective uncertainty.

Proposition 8. *If $\mathcal{Z} := \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^{n_x}\times\mathbb{R}^{n_y} \,|\, \boldsymbol{x}\in\mathcal{X}, A\boldsymbol{x}+B\boldsymbol{y} \leq \boldsymbol{\psi}\}$ is a simplex set, then affine decision rules are optimal in the TSLRO reformulation of the WCARM, when assumptions 1, 2, 3, and 5 hold, and WCRRM problems, when assumptions 1, 5, 6, and 7 hold, with objective uncertainty, i.e. problem* (20) *and* (29) *respectively.*

Proof. Considering the case of the WCARM problem, we start by establishing a second equivalent TSLRO reformulation for problem (4). In particular, for any fixed $\boldsymbol{x}$, we can let

$$
\begin{aligned}
h_1(\boldsymbol{x}) :=& \inf_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} \; \sup_{(\boldsymbol{\lambda},\boldsymbol{\gamma})\in\Upsilon_1(\boldsymbol{\zeta})} \; -\boldsymbol{\psi}^T\boldsymbol{\lambda} - \boldsymbol{v}^T\boldsymbol{\gamma} + \boldsymbol{c}^T\boldsymbol{x} + (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \\
=& \inf_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} \; \inf_{\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}'(\boldsymbol{x}')} \; -\boldsymbol{c}^T\boldsymbol{x}' - \boldsymbol{d}(\boldsymbol{\zeta})^T\boldsymbol{y}' + \boldsymbol{c}^T\boldsymbol{x} + (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \\
=& \inf_{\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}'(\boldsymbol{x}')} \; \sup_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x}),\boldsymbol{\lambda}\in\mathcal{L}(\boldsymbol{y},\boldsymbol{y}')} \; \boldsymbol{c}^T(\boldsymbol{x}-\boldsymbol{x}') + \boldsymbol{d}^T(\boldsymbol{y}-\boldsymbol{y}') - \boldsymbol{q}^T\boldsymbol{\lambda}\,,
\end{aligned}
$$

where $\mathcal{L}(\boldsymbol{y},\boldsymbol{y}') := \{\boldsymbol{\lambda}\in\mathbb{R}^s_+ \,|\, P^T\boldsymbol{\lambda} = D^T(\boldsymbol{y}'-\boldsymbol{y})\}$ and where we exploited strong duality of

$$
\inf_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} \; (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} - \boldsymbol{d}(\boldsymbol{\zeta})^T\boldsymbol{y}' \;=\; \sup_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x}),\boldsymbol{\lambda}\in\mathcal{L}(\boldsymbol{y},\boldsymbol{y}')} \; \boldsymbol{d}^T(\boldsymbol{y}-\boldsymbol{y}') - \boldsymbol{q}^T\boldsymbol{\lambda}\,.
$$

Note that strong duality follows from Assumption 3 for the same reasons as in the case of problem (14) (see proof of Proposition 1). Hence, our analysis gives rise to a dual reformulation for TSLRO (20).

In Bertsimas and de Ruiter (2016), it was established (see Theorem 2) that the conservative approximation obtained by employing affine decision rules on a TSLRO problem is exactly equivalent to the approximation obtained by employing affine decision rules on its dual reformulation. This implies that:

$$h_2(\boldsymbol{x}) := \sup_{(\boldsymbol{\lambda}(\cdot),\boldsymbol{\gamma}(\cdot))\in\Upsilon_1^{\text{aff}}} \inf_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} -\boldsymbol{\psi}^T\boldsymbol{\lambda}(\boldsymbol{\zeta},\boldsymbol{\rho}) - \boldsymbol{v}^T\boldsymbol{\gamma}(\boldsymbol{\zeta},\boldsymbol{\rho}) + \boldsymbol{c}^T\boldsymbol{x} + (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \tag{41}$$

$$= \sup_{\boldsymbol{y}(\cdot)\in\mathcal{Y}^{\text{aff}}(\boldsymbol{x}),\boldsymbol{\lambda}(\cdot)\in\mathcal{L}^{\text{aff}}(\boldsymbol{y}(\cdot))} \min_{\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}'(\boldsymbol{x}')} \boldsymbol{c}^T(\boldsymbol{x}-\boldsymbol{x}') + \boldsymbol{d}^T(\boldsymbol{y}(\boldsymbol{x}',\boldsymbol{y}') - \boldsymbol{y}') - \boldsymbol{q}^T\boldsymbol{\lambda}(\boldsymbol{x}',\boldsymbol{y}') \tag{42}$$

$$= \inf_{\boldsymbol{x}'\in\mathcal{X},\boldsymbol{y}'\in\mathcal{Y}'(\boldsymbol{x})} \sup_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x}),\boldsymbol{\lambda}\in\mathcal{L}(\boldsymbol{y},\boldsymbol{y}')} \boldsymbol{c}^T(\boldsymbol{x}-\boldsymbol{x}') + \boldsymbol{d}^T(\boldsymbol{y}-\boldsymbol{y}') - \boldsymbol{q}^T\boldsymbol{\lambda} \tag{43}$$

$$= h_1(\boldsymbol{x}),$$

where Υ_1^{aff} captures the set of all affine mappings for $\boldsymbol{\lambda} : \mathbb{R}^{n_\zeta}\times\mathbb{R}^s \to \mathbb{R}^m$ and $\boldsymbol{\gamma} : \mathbb{R}^{n_\zeta}\times\mathbb{R}^s \to \mathbb{R}^r$ such that $(\boldsymbol{\lambda}(\boldsymbol{\zeta},\boldsymbol{\rho}),\boldsymbol{\gamma}(\boldsymbol{\zeta},\boldsymbol{\rho})) \in \Upsilon_1(\boldsymbol{\zeta})$ for all $\boldsymbol{\zeta}\in\mathcal{U}$ and $\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})$, $\mathcal{Y}^{\text{aff}}(\boldsymbol{x})$ captures the affine mappings $\boldsymbol{y} : \mathbb{R}^{n_x}\times\mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ such that $\boldsymbol{y}(\boldsymbol{x}',\boldsymbol{y}') \in \mathcal{Y}(\boldsymbol{x})$ for all $\boldsymbol{x}'\in\mathcal{X}$ and $\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}')$, and $\mathcal{L}^{\text{aff}}(\boldsymbol{y}(\cdot))$ captures the affine mappings $\boldsymbol{\lambda} : \mathbb{R}^{n_x}\times\mathbb{R}^{n_y} \to \mathbb{R}^s$ such that $\boldsymbol{\lambda}(\boldsymbol{x}',\boldsymbol{y}') \in \mathcal{L}(\boldsymbol{y}(\boldsymbol{x}',\boldsymbol{y}'),\boldsymbol{y}')$ for all $\boldsymbol{x}'\in\mathcal{X}$ and $\boldsymbol{y}'\in\mathcal{Y}(\boldsymbol{x}')$. Specifically, while the equivalence between expression (41) and (42) follows from Theorem 2 of Bertsimas and de Ruiter (2016), the equivalence between (42) and (43) rather follows from Bertsimas and Goyal (2012) as exploited in the proof of Proposition 7.

In the case of WCRRM, the steps are very similar to the ones used in proving Proposition 7. We first let, for any fixed feasible u and $\boldsymbol{z}$ and associated $\boldsymbol{x} := u\boldsymbol{z} \in \mathcal{X}$ and $t := 1 - 1/u$, $h_1(u,\boldsymbol{z})$ stand for

$$h_1(u,\boldsymbol{z}) := \sup_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} h''(\boldsymbol{z},u,\boldsymbol{\zeta},\boldsymbol{\rho}),$$

where $h''(\boldsymbol{z},u,\boldsymbol{\zeta},\boldsymbol{\rho})$ is as defined in equation (34). Furthermore, we let $h_2(u,\boldsymbol{z})$ be the upper bound achieved when using affine decision rules for $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$. We must then have that:

$$\begin{aligned}
h_2(u,\boldsymbol{z}) = &\inf_{(\boldsymbol{\lambda}(\cdot),\boldsymbol{\gamma}(\cdot))\in\Upsilon_1^{\text{aff}}} u \\
&\quad\text{s.t.}\quad \boldsymbol{\psi}^T\boldsymbol{\lambda}(\boldsymbol{\zeta},\boldsymbol{\rho}) + \boldsymbol{v}^T\boldsymbol{\gamma}(\boldsymbol{\zeta},\boldsymbol{\rho}) - \boldsymbol{c}^T\boldsymbol{z} - (\boldsymbol{\psi}u - A\boldsymbol{z})^T\boldsymbol{\rho} \leq 0,\ \forall\,\boldsymbol{\zeta}\in\mathcal{U},\ \boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta}) \\
= &\inf_{(\boldsymbol{\lambda}(\cdot),\boldsymbol{\gamma}(\cdot))\in\Upsilon_1^{\text{aff}}} u \\
&\quad\text{s.t.}\quad \frac{1}{u}\boldsymbol{\psi}^T\boldsymbol{\lambda}(\boldsymbol{\zeta},\boldsymbol{\rho}) + \frac{1}{u}\boldsymbol{v}^T\boldsymbol{\gamma}(\boldsymbol{\zeta},\boldsymbol{\rho}) - \frac{1}{u}\boldsymbol{c}^T\boldsymbol{z} - (\boldsymbol{\psi} - \frac{1}{u}A\boldsymbol{z})^T\boldsymbol{\rho} \leq 0,\ \forall\,\boldsymbol{\zeta}\in\mathcal{U},\ \boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta}) \\
= &\begin{cases} u & \text{if } \inf_{(\boldsymbol{\lambda}(\cdot),\boldsymbol{\gamma}(\cdot))\in\Upsilon_1^{\text{aff}}} \sup_{\boldsymbol{\zeta}\in\mathcal{U},\,\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} \frac{1}{u}\boldsymbol{\psi}^T\boldsymbol{\lambda}(\boldsymbol{\zeta},\boldsymbol{\rho}) + \frac{1}{u}\boldsymbol{v}^T\boldsymbol{\gamma}(\boldsymbol{\zeta},\boldsymbol{\rho}) - \boldsymbol{c}^T\boldsymbol{x} - (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \leq 0 \\ \infty & \text{otherwise} \end{cases} \\
= &\begin{cases} u & \text{if } \sup_{\boldsymbol{\zeta}\in\mathcal{U},\,\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} \inf_{(\boldsymbol{\lambda},\boldsymbol{\gamma})\in\Upsilon_1(\boldsymbol{\zeta})} \frac{1}{u}\boldsymbol{\psi}^T\boldsymbol{\lambda} + \frac{1}{u}\boldsymbol{v}^T\boldsymbol{\gamma} - \boldsymbol{c}^T\boldsymbol{x} - (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \leq 0 \\ \infty & \text{otherwise} \end{cases} \\
= &\sup_{\boldsymbol{\zeta}\in\mathcal{U},\,\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} \inf_{(\boldsymbol{\lambda},\boldsymbol{\gamma})\in\Upsilon_1(\boldsymbol{\zeta})} u \\
&\quad\text{s.t.}\quad \boldsymbol{\psi}^T\boldsymbol{\lambda}(\boldsymbol{\zeta},\boldsymbol{\rho}) + \boldsymbol{v}^T\boldsymbol{\gamma}(\boldsymbol{\zeta},\boldsymbol{\rho}) - \boldsymbol{c}^T\boldsymbol{z} - (\boldsymbol{\psi}u - A\boldsymbol{z})^T\boldsymbol{\rho} \leq 0 \\
= &h_1(u,\boldsymbol{z}),
\end{aligned}$$

Note that again here we exploit the fact that affine decision rules on

$$\sup_{\boldsymbol{\zeta}\in\mathcal{U},\,\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})}\ \inf_{(\boldsymbol{\lambda},\gamma)\in\Upsilon_1(\boldsymbol{\zeta})}\ \frac{1}{u}\boldsymbol{\psi}^T\boldsymbol{\lambda}+\frac{1}{u}\boldsymbol{v}^T\boldsymbol{\gamma}-\boldsymbol{c}^T\boldsymbol{x}-(\boldsymbol{\psi}-A\boldsymbol{x})^T\boldsymbol{\rho}$$

gives the same optimal value as using it on

$$\inf_{\boldsymbol{x}',\boldsymbol{y}':u\boldsymbol{x}'\in\mathcal{X},u\boldsymbol{y}'\in\mathcal{Y}(u\boldsymbol{x}')}\ \sup_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x}),\boldsymbol{\lambda}\in\mathcal{L}(\boldsymbol{y},\boldsymbol{y}')}\ \boldsymbol{c}^T(\boldsymbol{x}-\boldsymbol{x}')+\boldsymbol{d}^T(\boldsymbol{y}-\boldsymbol{y}')-\boldsymbol{q}^T\boldsymbol{\lambda}$$

which is its dual reformulation and for which we can verify that the set $\{(\boldsymbol{x}',\boldsymbol{y}')\,|\,u\boldsymbol{x}'\in\mathcal{X},u\boldsymbol{y}'\in\mathcal{Y}(u\boldsymbol{x}')\}$ is a simplex set when $\{(\boldsymbol{x},\boldsymbol{y})\,|\,\boldsymbol{x}\in\mathcal{X},\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x})\}$ is one. Hence, according to Theorem 2 in Bertsimas and de Ruiter (2016) and Theorem 1 in Bertsimas and Goyal (2012), affine decision rules must be optimal in both cases. □

Interestingly, Proposition 8 provides a polynomial time reformulation for the WCARM and WCRRM versions of the continuous knapsack problem.

Corollary 9. *The linear program obtained by employing affine decision rules on the TSLRO reformulation of the WCARM problem*

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}}\ \sup_{\boldsymbol{\zeta}\in\mathcal{U}}\,(\max_{\boldsymbol{x}'\in\mathcal{X}}\boldsymbol{d}(\boldsymbol{\zeta})^T\boldsymbol{x}-\boldsymbol{d}(\boldsymbol{\zeta})^T\boldsymbol{x})\,,$$

where $\mathcal{X}:=\{\boldsymbol{x}\in\mathbb{R}_+^{n_x}\,|\,\boldsymbol{w}^T\boldsymbol{x}\le v\}$ with $\boldsymbol{w}\in\mathbb{R}_+^{n_x}$ and $v\in\mathbb{R}_+$, is exact for all polyhedral uncertainty set $\mathcal{U}$, and similarly for the WCRRM version of this problem given that the assumptions described in Proposition 8 hold.

This corollary extends the result in Averbakh (2004) which identified a $O(n_x\log(n_x))$ time algorithm for the WCARM version of the knapsack problem under interval uncertainty.

Following the work of Ardestani-Jaafari and Delage (2016), the result presented in Proposition 7 can be extended to other form of uncertainty sets in the case that $h(\boldsymbol{x},\boldsymbol{\zeta})$ captures the sum of piecewise linear concave functions.

Proposition 10. *If $h(\boldsymbol{x},\boldsymbol{\zeta})$ is a sum of of piecewise linear concave functions of the form:*

$$h(\boldsymbol{x},\boldsymbol{\zeta}):=\sum_{i=1}^{N}\min_{k=1,\ldots,K}\boldsymbol{\alpha}_{ik}(\boldsymbol{x})^T\boldsymbol{\zeta}+\beta_{ik}(\boldsymbol{x})=\begin{array}{ll}\max\limits_{\boldsymbol{y}} & \sum\limits_{i=1}^{n_y}y_i\\ \text{s.t.} & y_i\le\boldsymbol{\alpha}_{ik}(\boldsymbol{x})^T\boldsymbol{\zeta}+\beta_{ik}(\boldsymbol{x}),\,\forall\,i,\,\forall\,k\,,\end{array}\tag{44}$$

for some affine mappings $\boldsymbol{\alpha}_{ik}:\mathbb{R}^{n_x}\to\mathbb{R}^{n_\zeta}$ and $\beta_{ik}:\mathbb{R}^{n_x}\to\mathbb{R}$, the uncertainty set $\mathcal{U}$ is the budgeted uncertainty set

$$\mathcal{U}:=\{\boldsymbol{\zeta}\in\mathbb{R}^{n_\zeta}\,|\,\exists\,\boldsymbol{\zeta}^+\in\mathbb{R}_+^{n_\zeta},\,\exists\,\boldsymbol{\zeta}^-\in\mathbb{R}_+^{n_\zeta},\,\boldsymbol{\zeta}=\boldsymbol{\zeta}^+-\boldsymbol{\zeta}^-,\,\boldsymbol{\zeta}^++\boldsymbol{\zeta}^-\le 1\,,\,\sum_i\zeta_i^++\zeta_i^-=\Gamma\}\,,\tag{45}$$

and the following conditions are satisfied:

1. *Either of the following applies:*
   - *i.* $\Gamma=1$
   - *ii.* $\Gamma=n_\zeta$ *and uncertainty is "additive": i.e.* $\boldsymbol{\alpha}_{ik}(\boldsymbol{x})=\bar{\alpha}_{ik}(\boldsymbol{x})(\sum_{\ell<i}\hat{\alpha}_\ell(\boldsymbol{x})\boldsymbol{e}_\ell)$ *for some* $\bar{\alpha}_{ik}:\mathbb{R}^{n_x}\to\mathbb{R}$ *for all i and k and some* $\hat{\boldsymbol{\alpha}}:\mathbb{R}^{n_x}\to\mathbb{R}^{n_\zeta}$
   - *iii.* Γ *is integer and the objective function is "decomposable": i.e.* $\boldsymbol{\alpha}_{ik}(\boldsymbol{x})=\bar{\alpha}_{ik}(\boldsymbol{x})\boldsymbol{e}_i$ *for some* $\bar{\alpha}_{ik}:\mathbb{R}^{n_x}\to\mathbb{R}$ *for all i and k*
2. $\max_{\boldsymbol{x}\in\mathcal{X}}h(\boldsymbol{x},\boldsymbol{\zeta})=\boldsymbol{\gamma}^T\boldsymbol{\zeta}+\bar{\gamma}$ *for some* $\boldsymbol{\gamma}\in\mathbb{R}^{n_\zeta}$ *and* $\bar{\gamma}\in\mathbb{R}$

Then, affine decision rules with respect to $(\boldsymbol{\zeta}^+, \boldsymbol{\zeta}^-, \boldsymbol{x}', \boldsymbol{y}')$ are optimal in the TSLRO reformulation of the WCARM and WCRRM problems, i.e. problem (16) *and* (26) *respectively.*

The proof proceeds in two steps. The first step consists in extending Corollary 1 in Ardestani-Jaafari and Delage (2016) to the following formulation:

$$\underset{x\in\mathcal{X}}{\text{maximize}} \quad \min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} h(\boldsymbol{x}, \boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) - \bar{\gamma} - \boldsymbol{\gamma}^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-)\,, \tag{46}$$

where

$$\mathcal{U}_\pm(\Gamma) := \{(\boldsymbol{\zeta}^+, \boldsymbol{\zeta}^-) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \,|\, \boldsymbol{\zeta}^+ + \boldsymbol{\zeta}^- \leq 1,\, \sum_i \boldsymbol{\zeta}^+_i + \boldsymbol{\zeta}^-_i = \Gamma\}\,,$$

and where $h(\boldsymbol{x}, \boldsymbol{\zeta})$ is a sum of piecewise linear concave functions as defined in (44). Namely, that affine decision rules are optimal for problem (46) when $h(\boldsymbol{x}, \boldsymbol{\zeta})$ and Γ satisfy one of the three conditions described in our proposition. This can then be used to demonstrate that they are optimal for problem (16) and (26) following the same arguments as those used in the proof of Proposition 7 where the equivalence between (35) and (38), and between (39) and (40) is now supported by what was established in the first step. For sake of conciseness, we focus on the first step which details are presented in Appendix C.

Propositions 10 and 7 effectively extend the set of problem classes for which a polynomial time solution scheme is known. In particular, it extends the results of Vairaktarakis (2000) for multi-item newsvendor problems to include simplex sets and budgeted uncertainty sets with integer budget. They similarly provides a polynomial time solution scheme for a large class of lot-sizing problems under the budgeted uncertainty set as long as $\Gamma = 1$ or n_ζ. Unlike in the work of Vairaktarakis (2000) and Z. (2011), tractability does not come from exploiting specifically designed algorithms for each of these application but is rather simply achieved by employing the general linear decision rules approach on the TSLRO reformulation. It further naturally serves as theoretical evidence of the effectiveness of such an approach for general regret minimization.

7 Numerical Results

In this section, we evaluate the numerical performance of exact and approximate solution schemes that are commonly used to solve two-stage linear robust optimization problems when employed to solve the TSLRO reformulations of worst-case regret minimization problems. This is done in the context of two representative applications of TSLRO, namely a multi-item newsvendor problem and a production-transportation problem, which are respectively special cases of TSLRO with right hand side uncertainty and objective uncertainty. Our objective consists in comparing both the solution time and quality of first stage decisions that are obtained using exact and approximate methods and provide empirical evidence regarding whether two-stage regret minimization problems are more difficult to solve than their robust optimization version.

While a number of approximation schemes from the adjustable robust optimization literature could be put to the test, we focus our analysis on the AARC approximation method described in Section (3.1.2). Similarly, we rely on the C&CG method presented in Section (3.1.1) to solve the TSLRO reformulations of WCARM problems exactly, and on the column and constraint generation algorithm of Ayoub and Poss (2016), called C&CG*, for WCRRM problems. A time limit of 4 hours ($14,400$ seconds) and optimality tolerance of 10^{-6} are imposed on all solution schemes. The quality of the AARC approximation scheme is reported in terms of relative optimality gap (in %) in the case of a WCARM model, and absolute optimality gap for WCRRM models since the objective function is already expressed in percentage. All algorithms were implemented in MATLAB R2017b using the YALMIP toolbox and CPLEX 12.8.0 as the solver for all linear programming models.

7.1 Multi-item newsvendor problem

The first application that we consider is the multi-item newsvendor problem which was studied in its robust optimization form in Ardestani-Jaafari and Delage (2016) and Ardestani-Jaafari and Delage (2017). The single-stage robust formulation of this problem is as follows:

$$\underset{\boldsymbol{x}\geq 0}{\text{maximize}} \quad \min_{\boldsymbol{\zeta}\in\mathcal{U}} \sum_{i=1}^{n_y} p_i \min(x_i, \zeta_i) - c_i x_i + s_i \max(x_i - \zeta_i, 0) - b_i \max(\zeta_i - x_i, 0)\,, \quad (47)$$

where $p_i \geq 0$, $c_i \in [0, p_i]$, $s_i \in [0, c_i]$, and $b_i \geq 0$ represent sale price, ordering cost, salvage value, and shortage cost of a unit of item i, with $i = 1, \ldots, n_y$, respectively. Decision variable x_i is the initial ordering amount of item i. We consider two forms of uncertainty sets, which respectively model the fact that the demand for each item is assumed correlated or not. The "uncorrelated demand" uncertainty set is defined straightforwardly in terms of the well known budgeted set (see Bertsimas and Sim 2004):

$$\mathcal{U}(\Gamma) = \left\{ \boldsymbol{\zeta} \;\middle|\; \exists \boldsymbol{\delta}^+, \boldsymbol{\delta}^- \in \mathbb{R}^{n_y}, \begin{array}{l} \boldsymbol{\delta}^+ \geq 0,\, \boldsymbol{\delta}^- \geq 0 \\ \delta_i^+ + \delta_i^- \leq 1,\, \forall i = 1, \ldots, n_y \\ \sum_{i=1}^{n_y} \delta_i^+ + \delta_i^- = \Gamma \\ \zeta_i = \bar{\zeta}_i + \hat{\zeta}_i(\delta_i^+ - \delta_i^-),\, \forall i = 1, \ldots, n_y \end{array} \right\},$$

where $\bar{\zeta}_i$ and $\hat{\zeta}_i$ denote the nominal demand and the maximum demand deviation of the item i and Γ captures a budget of maximum number of deviations from the nominal demand. We also consider a "correlated demand" uncertainty set defined as follows:

$$\tilde{\mathcal{U}}(\Gamma) = \left\{ \boldsymbol{\zeta} \;\middle|\; \exists \boldsymbol{\delta}^+, \boldsymbol{\delta}^- \in \mathbb{R}^{n_y}, \begin{array}{l} \boldsymbol{\delta}^+ \geq 0,\, \boldsymbol{\delta}^- \geq 0 \\ \delta_i^+ + \delta_i^- \leq 1,\, \forall i = 1, \ldots, n_y \\ \sum_{i=1}^{n_y} \delta_i^+ + \delta_i^- = \Gamma \\ \zeta_i = \bar{\zeta}_i + \hat{\zeta}_i(\delta_{j_1(i)}^+ + \delta_{j_2(i)}^+ - \delta_{j_1(i)}^- - \delta_{j_2(i)}^-)/2,\, \forall i = 1, \ldots, n_y \end{array} \right\},$$

where $j : \{1, \ldots, n_y\} \to \{1, \ldots, n_y\}^2$ identifies two sources of perturbation of item i such that items i_1 and i_2 are correlated if $j_{\ell_1}(i_1) = j_{\ell_2}(i_2)$ for some $(\ell_1, \ell_2) \in \{1, 2\}^2$.

We consider three different sizes of the problem, namely $n_y \in \{5, 10, 20\}$. For each size, we generate 10 problem instances randomly according to the following procedure. Each sale price p_i is uniformly and independently generated on the interval $[0.5,\, 1]$, each ordering cost c_i uniformly generated on $[0.3p_i,\, 0.9p_i]$, and salvage value s_i and shortage cost b_i are drawn uniformly at random from $[0.1c_i,\, c_i]$. Nominal demand $\bar{d}_i = 10$ for each item i while the maximum demand perturbation is generated uniformly on $[0.3\bar{d}_i,\, 0.6\bar{d}_i]$. In the case of the correlated uncertainty set $\tilde{\mathcal{U}}$, for each item i the pair $(j_1(i),\, j_2(i))$ is drawn randomly among all possible pairs such that $j_1(i) \neq j_2(i)$. The budget Γ is fixed among the levels $\Gamma \in \{0.3n_y, 0.5n_y, 0.7n_y, n_y\}$. Tables 3 and 4 present the average performance of C&CG and AARC in solving the classical robust optimization, the worst-case absolute regret minimization and the worst-case relative regret minimization formulation when accounting for the uncorrelated and correlated uncertainty sets respectively.

Looking at Table 3, one can remark that for the instances with $n_y = 5$ items, the average optimality gaps achieved by the AARC approach are of similar small sizes in the case of classical robust optimization as for worst-case regret minimization. The optimality gap is also surprisingly small (below 0.3%) for the WCRRM problems. Since the instances studied in this table employ an uncorrelated uncertainty set, the empirical evidence confirms the findings of Proposition 10 which states that, similarly as for the robust optimization formulation (see Ardestani-Jaafari and Delage 2016), AARC provides exact solutions for WCARM and WCRRM when Γ is integer since the objective function is decomposable in $\boldsymbol{\delta}^+ - \boldsymbol{\delta}^-$. This applies for all cases where $\Gamma = n_y$ and all cases where $n_y \in \{10,\, 20\}$ since all Γ's are on deciles of n_y.

Table 3: Multi-item Newsvendor Problem - Uncorrelated Uncertainty Set

Problem Size	Decision Criterion	Result Description	Level of Uncertainty (in % of n_y)			
			30%	50%	70%	100%
5 items	Worst-case Profit (RO)	Avg Rel. Gap - AARC	0.72%	0.62%	0.92%	0.00%
		Avg CPU time (s) - AARC	1.3	1.3	1.3	1.3
		Avg CPU time (s) - C&CG	71.8	119.9	148.7	85.8
	Worst-case Absolute Regret (WCARM)	Avg Rel. Gap - AARC	2.03%	0.49%	0.14%	0.00%
		Avg CPU time (s) - AARC	1.3	1.3	1.3	1.3
		Avg CPU time (s) - C&CG	116.7	143.1	105.8	82.5
	Worst-case Relative Regret (WCRRM)	Avg Abs. Gap - AARC	0.24%	0.15%	0.08%	0.00%
		Avg CPU time (s) - AARC	0.2	0.2	0.3	0.3
		Avg CPU time (s) - C&CG*	142.7	154.1	166.8	118.4
10 items	Worst-case Profit (RO)	Avg Rel. Gap - AARC	0.00%	0.00%	0.00%	0.00%
		Avg CPU time (s) - AARC	1.4	1.4	1.5	1.5
		Avg CPU time (s) - C&CG	96.9	138.5	282.6	174.8
	Worst-case Absolute Regret (WCARM)	Avg Rel. Gap - AARC	0.00%	0.00%	0.00%	0.00%
		Avg CPU time (s) - AARC	1.5	1.5	1.5	1.5
		Avg CPU time (s) - C&CG	184.0	239.4	201.8	153.1
	Worst-case Relative Regret (WCRRM)	Avg Abs. Gap	0.00%	0.00%	0.00%	0.00%
		Avg CPU time (s) - AARC	0.4	0.4	0.4	0.4
		Avg CPU time (s) - C&CG*	238.5	315.0	312.6	206.2
20 items	Worst-case Profit (RO)	Avg Rel. Gap - AARC	0.00%	0.00%	0.00%	0.00%
		Avg CPU time (s) - AARC	1.9	1.9	2.0	2.1
		Avg CPU time (s) - C&CG	227.3	381.9	649.8	460.3
	Worst-case Absolute Regret (WCARM)	Avg Rel. Gap - AARC	0.00%	0.00%	0.00%	0.00%
		Avg CPU time (s) - AARC	2.0	2.1	2.2	2.2
		Avg CPU time (s) - C&CG	494.7	760.6	781.3	367.7
	Worst-case Relative Regret (WCRRM)	Avg Abs. Gap - AARC	0.00%	0.00%	0.00%	0.00%
		Avg CPU time (s) - AARC	1.0	1.0	1.1	1.3
		Avg CPU time (s) - C&CG*	891.1	7,528.4 [6]	— [0]	5,115.3

[ ] Indicates the number of instances solved by C&CG* algorithm within the 4 hours time limit. In this case, the average is computed on the instance that were solved to optimality within the time limit.

When it comes to comparing computation times, one may make three interesting observations. First, all AARC approximation models are solved in less then 3 seconds (on average) which is more than one order of magnitude faster than the time needed to solve any of these problems using C&CG. This can be explained by the well known fact that each step of C&CG involves solving an NP-hard mixed integer linear program. Secondly, it appears to be generally true that both of the AARC and C&CG solution schemes have a similar runtime whether they are used to solve the RO model or the WCARM. This seems to support the claim that regret minimization has the same complexity as robust optimization for two-stage linear program with right-hand side uncertainty. On the other hand, it also appears that the C&CG* approach used for WCRRM leads to longer run times than what is needed for RO models. Finally, we see that in the case of $n_y = 20$ the C&CG* scheme is unable to solve a number of problem instances within the allocated time for $\Gamma = 10$ and 14. This is in sharp contrast with the AARC approach which identifies optimal solutions in less than a couple of seconds. This evidence reinforces the idea that modern approximation methods that exist for RO models can provide high performance algorithms for regret minimization problems.

Looking at Table 4 where problem instances have correlated demand, we draw similar conclusions as with Table 3. Namely, we observe that AARC provides optimal solution when $\Gamma = n_y$,

Table 4: Multi-item Newsvendor Problem - Correlated Uncertainty Set

Problem Size	Decision Criterion	Result Description	Level of Uncertainty (in % of n_y) 30%	50%	70%	100%
5 items	Worst-case Profit (RO)	Avg Rel. Gap - AARC	1.46%	3.11%	2.39%	0.00%
		Avg CPU time (s) - AARC	1.3	1.3	1.3	1.3
		Avg CPU time (s) - C&CG	81.6	86.6	93.2	73.7
	Worst-case Absolute Regret (WCARM)	Avg Rel. Gap - AARC	3.58%	3.68%	1.61%	0.00%
		Avg CPU time (s) - AARC	1.3	1.3	1.3	1.3
		Avg CPU time (s) - C&CG	78.9	83.0	96.8	75.1
	Worst-case Relative Regret (WCRRM)	Avg Abs. Gap - AARC	0.68%	0.84%	0.69%	0.00%
		Avg CPU time (s) - AARC	0.2	0.2	0.2	0.2
		Avg CPU time (s) - C&CG*	103.3	118.9	132.1	93.3
10 items	Worst-case Profit (RO)	Avg Rel. Gap - AARC	1.30%	1.62%	0.62%	0.00%
		Avg CPU time (s) - AARC	1.4	1.4	1.4	1.4
		Avg CPU time (s) - C&CG	115.9	145.9	178.4	126.4
	Worst-case Absolute Regret (WCARM)	Avg Rel. Gap - AARC	3.16%	0.87%	0.16%	0.00%
		Avg CPU time (s) - AARC	1.4	1.4	1.4	1.4
		Avg CPU time (s) - C&CG	135.9	177.5	164.6	120.7
	Worst-case Relative Regret (WCRRM)	Avg Abs. Gap - AARC	0.30%	0.13%	0.09%	0.00%
		Avg CPU time (s) - AARC	0.3	0.3	0.3	0.4
		Avg CPU time (s) - C&CG*	208.3	262.0	258.9	157.8
20 items	Worst-case Profit (RO)	Avg Rel. Gap - AARC	0.62%	0.52%	0.10%	0.00%
		Avg CPU time (s) - AARC	1.7	1.7	1.8	1.9
		Avg CPU time (s) - C&CG	286.3	451.7	582.1	314.3
	Worst-case Absolute Regret (WCARM)	Avg Rel. Gap - AARC	0.66%	0.05%	0.01%	0.00%
		Avg CPU time (s) - AARC	1.9	2.0	2.0	2.1
		Avg CPU time (s) - C&CG	428.3	576.6	500.8	248.5
	Worst-case Relative Regret (WCRRM)	Avg Abs. Gap - AARC	0.07%	0.05%	0.02%	0.00%
		Avg CPU time (s) - AARC	0.8	0.9	0.9	1.0
		Avg CPU time (s) - C&CG*	717.1	2,681.5	6,287.9	567.8

which might indicate that there are other conditions than those identified in Section 6 where affine decision rules are optimal. For other cases, the quality of approximation is very high overall all versions of the problems, showing off with a maximum average gap of 3.68% and 0.84% for the WCARM and WCRRM problems respectively. In terms of the run times, the observations are also similar except for the instances where $n_y = 20$ which appear to be less challenging for the C&CG* scheme then they were when demand was uncorrelated. Indeed, C&CG* is able here to converge to an optimal solution within the time limit for all instances although this could simply be due to the specific structure of the 10 instances that were drawn for this part of the study. Overall, this study seem to indicate that AARC is a much more favorable approach for tackling larger scale regret minimization problems.

7.2 Production-transportation problem

Our second application consists of the production-transportation problem with uncertainty in transportation cost considered in Bertsimas et al. (2010). Specifically, in this problem one considers m facilities and n customer locations. Each facility has a production capacity of $\bar{x}_i$ goods. The units produced at these facilities should be shipped to the customer locations in order to cover a predefined set of orders. The difficulty for the manager resides in the fact that transportation costs are unknown when production decisions are made. The corresponding

TSLRO problem can be defined as follows:

$$\begin{aligned}
\underset{0\leq \boldsymbol{x}\leq \bar{\boldsymbol{x}},\, y(\boldsymbol{\zeta})}{\text{minimize}}\ \max_{\boldsymbol{\zeta}\in\mathcal{U}}\quad & \sum_{i=1}^{m} c_i x_i + \sum_{i=1}^{m}\sum_{j=1}^{n} \zeta_{ij} y_{ij}(\boldsymbol{\zeta}) && \text{(48a)}\\
\text{subject to}\quad & \sum_{i=1}^{m} y_{ij}(\boldsymbol{\zeta}) = d_j,\ \forall j\in\mathcal{J},\ \forall\boldsymbol{\zeta}\in\mathcal{U} && \text{(48b)}\\
& \sum_{j=1}^{n} y_{ij}(\boldsymbol{\zeta}) = x_i,\ \forall i\in\mathcal{I},\ \forall\boldsymbol{\zeta}\in\mathcal{U} && \text{(48c)}\\
& y(\boldsymbol{\zeta}) \geq 0,\ \forall\boldsymbol{\zeta}\in\mathcal{U}, && \text{(48d)}
\end{aligned}$$

where for each facility location i, c_i is the production cost, while for each customer location j, d_j refers to the demand that needs to be covered, and $\boldsymbol{\zeta}_{ij}$ is the initially unknown transportation cost per unit from production facility i to customer location j. This problem has two-stages of decisions, namely the here-and-now production decisions $\boldsymbol{x}$, and the wait-and-see transportation decisions $\boldsymbol{y}$, which are made once transportation costs are observed. Finally, we define the uncertainty set as

$$\mathcal{U}(\Gamma) = \left\{ \boldsymbol{\zeta} \,\middle|\, \exists \boldsymbol{\delta}^+, \boldsymbol{\delta}^- \in \mathbb{R}^m,\ \begin{array}{l} \boldsymbol{\delta}^+ \geq 0,\ \boldsymbol{\delta}^- \geq 0 \\ \delta_i^+ + \delta_i^- \leq 1,\ \forall i\in\mathcal{I} \\ \sum_i \delta_i^+ + \delta_i^- = \Gamma \end{array},\ \zeta_{ij} = \bar{\zeta}_{ij} + \hat{\zeta}_{ij}(\delta_i^+ - \delta_i^-),\ \forall i\in\mathcal{I},\ \forall j\in\mathcal{J} \right\},$$

where $\bar{\zeta}_{ij}$ and $\hat{\zeta}_{ij}$ are respectively the nominal cost and maximum cost deviation for transporting each unit of good transported from i to j. Note that in defining $\mathcal{U}(\Gamma)$, we make the uncertainty about the costs for transports from the same facility entirely correlated which allows us to consider $\Gamma \in [0\,, m]$, although one could easily consider each transportation cost to be independent from each other.

In our numerical experiments, we consider three different sizes of the problem, namely $(m, n) \in \{(3, 6), (5, 10), (7, 14)\}$. In each case, we generate 10 instances randomly. To do so, we start by randomly generating $m + n$ locations within the unit square. The nominal transportation cost per unit from facility i to customer j is set to the Euclidean distance between their locations and the maximum perturbation of this cost is supposed to be 50% of the nominal value. The production costs are uniformly and independently generated on the interval $[0.5\frac{\sum_{ij}\bar{\zeta}_{ij}}{mn},\, 1.5\frac{\sum_{ij}\bar{\zeta}_{ij}}{mn}]$. We fix the production capacities $\bar{x}_i$ to one. Given that this leads to a maximum total production of m units, the size of each order d_i is uniformly generated on the interval $[0.5m/n,\, m/n]$. The empirical performance of all solution schemes on all three forms of problems with $\Gamma \in \{0.3m, 0.5m, 0.7m, m\}$ are presented in Table 5. Note that in the case of the RO model, as described in Section (3.2), one can easily identify an optimal solution by solving the so-called robust counterpart (RC) model which takes the form of a linear program.

Looking at Table 5, one remarks that the average of the optimality gaps achieved by the AARC approach for the WCARM model is always below 8% for all values of Γ and all problem sizes. This is a poorer performance than in the case of the multi-item news-vendor problem yet still makes the AARC approach attractive when comparing to the convergence time of C&CG for problems of size $m = 7$ and $n = 14$ where all AARC models where solved in less than 8 minutes while C&CG takes around near 2 hours. It is also obvious that the RO model is more tractable than WCARM and WCRRM due to the fact that uncertainty is limited to the objective function. Moreover, it appears that the WCRRM model is especially difficult to solve exactly in this setting while the AARC approach once again performs surprisingly well both in terms of computation time and quality of solutions. Indeed, the average absolute gap remained under $< 1\%$ for all categories of instances where exact solutions could be identified.

Table 5: Production-Transportation Problem

Problem Size	Decision Criterion	Result Description	Level of Uncertainty (in % of m) 30%	50%	70%	100%
3 facilities 6 customers	Worst-case Cost (RO)	Avg CPU time (s) - RC	0.7	0.8	0.9	0.9
	Worst-case Absolute Regret (WCARM)	Avg Rel. Gap - AARC	0.55%	2.28%	2.51%	4.07%
		Avg CPU time (s) - AARC	1.9	1.9	2.0	2.0
		Avg CPU time (s) - C&CG	8.5	10.4	10.6	11.0
	Worst-case Relative Regret (WCRRM)	Avg Abs. Gap - AARC	0.02%	0.19%	0.23%	0.55%
		Avg CPU time (s) - AARC	5.9	6.0	6.0	6.0
		Avg CPU time (s) - C&CG*	89.4	102.0	98.9	112.9
5 facilities 10 customers	Worst-case Cost (RO)	Avg CPU time (s) - RC	1.0	1.2	1.5	1.8
	Worst-case Absolute Regret (WCARM)	Avg Rel. Gap - AARC	6.71%	7.21%	5.68%	4.97%
		Avg CPU time (s) - AARC	19.7	19.6	19.4	20.2
		Avg CPU time (s) - C&CG	42.9	65.4	93.2	95.8
	Worst-case Relative Regret (WCRRM)	Avg Abs. Gap - AARC	0.39%	0.66%	0.78%	0.79%
		Avg CPU time (s) - AARC	23.3	24.6	23.9	23.9
		Avg CPU time (s) - C&CG*	299.7	555.4	1,000.3	1,564.0
7 facilities 14 customers	Worst-case Cost (RO)	Avg CPU time (s) - RC	3.0	3.9	5.0	6.0
	Worst-case Absolute Regret (WCARM)	Avg Rel. Gap - AARC	4.14%	4.54%	4.59%	4.21%
		Avg CPU time (s) - AARC	442.9	373.2	318.7	296.4
		Avg CPU time (s) - C&CG	3,425.4	8,365.1	6,967.8	7,468.7
	Worst-case Relative Regret (WCRRM)	Avg Abs. Gap - AARC	—	—	—	—
		Avg CPU time (s) - AARC	352.5	319.7	346.9	451.3
		Avg CPU time (s) - C&CG*	>14,400	>14,400	>14,400	>14,400

In order to shed more light on the difficulties of solving WCRRM for larger size problems, we present in Table 6 a description of the performance of both AARC and C&CG* for each of the 10 large problem instances for which C&CG* was unable to converge in less than 4 hours. In particular, the table shows that when $\Gamma = 0.3m$, for 3 out of 10 instances, the C&CG* algorithm is unable to provide the cuts needed to bound the minimal worst-case relative regret away from 0%. Furthermore, in instance #5, it is even unable to identify the most violated constraint in its first iteration within the allotted time. This phenomenon becomes more frequent at the size of Γ is increased. In the limit when $\Gamma = m$, 6 out of the 10 instances did not complete there first round of constraint generation because of the difficulty of the subproblem. For sake of completeness, we provide the bounds that can be computed on the optimality gap of AARC given the state of the C&CG* algorithm after four hours. Overall, these seem to support the idea that, for this class of problems, AARC is a valuable approximation scheme and that the design of efficient exact algorithms constitutes a promising direction of future research.

Acknowledgement

The authors gratefully acknowledge support from the Fonds de recherche du Qubec – Nature et technologies (FRQNT) [271693] and of the Canadian Natural Sciences and Engineering Research Council [Grant RGPIN-2016-05208 and 492997-2016]. They are also thankful to Shiva Zokaee for her involvement in a preliminary study involving a facility location problem, and valuable discussions that followed regarding possible extensions.

Table 6: WCRRM - Production-Transportation Problem with 7 Facilities and 14 Customers

Ins.	Uncertainty Level (in % of m)								
	30%			70%			100%		
	AARC UB	C&CG* LB	AARC Abs. gap	AARC UB	C&CG* LB	AARC Abs. gap	AARC UB	C&CG* LB	AARC Abs. gap
1	4.72%	0.00%	≤ 4.72%	7.05%	0.00%	≤ 7.05%	7.62%	0.00%	≤ 7.62%
2	4.14%	3.68%	≤ 0.45%	5.88%	0.00%*	≤ 5.88%	6.04%	0.00%	≤ 6.04%
3	2.78%	0.00%	≤ 2.78%	4.47%	0.00%	≤ 4.47%	4.77%	0.00%*	≤ 4.77%
4	4.30%	2.20%	≤ 2.10%	7.09%	0.00%*	≤ 7.09%	7.35%	0.00%	≤ 7.35%
5	3.05%	0.00%*	≤ 3.05%	4.54%	0.00%*	≤ 4.54%	4.64%	0.00%*	≤4.64%
6	3.79%	2.28%	≤ 1.51%	6.15%	0.00%	≤ 6.15%	6.45%	0.00%*	≤ 6.45%
7	3.64%	2.61%	≤ 1.03%	5.70%	0.00%*	≤5.70%	5.93%	0.00%*	≤ 5.93%
8	6.28%	1.40%	≤ 4.88%	9.67%	0.00%	≤ 9.67%	9.98%	0.00%	≤ 9.98%
9	4.94%	1.40%	≤ 3.54%	7.32%	0.00%	≤ 7.32%	7.67%	0.00%*	≤ 7.67%
10	2.57%	0.56%	≤ 2.01%	3.91%	0.00%	≤ 3.91%	4.13%	0.00%*	≤ 4.13%

* Indicates that C&CG* was unable to identify the most violated constraint within 4 hours in its first iteration.

References

Aissi, H., C. Bazgan, D. Vanderpooten. 2009. Min-max and min-max regret versions of combinatorial optimization problems: A survey. *European Journal of Operational Research* **197**(2) 427–438.

Ardestani-Jaafari, A., E. Delage. 2016. Robust optimization of sums of piecewise linear functions with application to inventory problems. *Operations Research* **64**(2) 474–494.

Ardestani-Jaafari, A., E. Delage. 2017. Linearized robust counterparts of two-stage robust optimization problem with applications in operations management. *Cahier du GERAD G-2016-69* .

Assavapokee, T., M. J. Realff, J. C. Ammons. 2008a. Min-max regret robust optimization approach on interval data uncertainty. *Journal of Optimization Theory and Applications* **137**(2) 297–316.

Assavapokee, T., M. J. Realff, J. C. Ammons, I.-H. Hong. 2008b. Scenario relaxation algorithm for finite scenario-based min-max regret and min-max relative regret robust optimization. *Computers & Operations Research* **35**(6) 2093 – 2102.

Audibert, J.-Y., S. Bubeck, G. Lugosi. 2014. Regret in online combinatorial optimization. *Mathematics of Operations Research* **39**(1) 31–45.

Averbakh, I. 2001. On the complexity of a class of combinatorial optimization problems with uncertainty. *Mathematical Programming* **90**(2) 263–272.

Averbakh, I. 2004. Minmax regret linear resource allocation problems. *Operations Research Letters* **32**(2) 174 – 180.

Averbakh, I., V. Lebedev. 2004. Interval data minmax regret network optimization problems. *Discrete Applied Mathematics* **138**(3) 289 – 301.

Averbakh, I., V. Lebedev. 2005. On the complexity of minmax regret linear programming. *European Journal of Operational Research* **160**(1) 227 – 231.

Ayoub, J., M. Poss. 2016. Decomposition for adjustable robust linear optimization subject to uncertainty polytope. *Computational Management Science* **13**(2) 219–239.

B., Dimitris, D. A. Iancu, P. A. Parrilo. 2010. Optimality of affine policies in multistage robust optimization. *Mathematics of Operations Research* **35**(2) 363–394.

Ben-Tal, A., L. El Ghaoui, A. Nemirovski. 2009. *Robust optimization*. Princeton University Press.

Ben-Tal, A., B. Golany, A. Nemirovski, J.-P. Vial. 2005. Retailer-supplier flexible commitments contracts: A robust optimization approach. *Manufacturing & Service Operations Management* **7**(3) 248–271.

Ben-Tal, A., A. Goryashko, E. Guslitzer, A. Nemirovski. 2004. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming* **99**(2) 351–376.

Bertsimas, D., F. J. C. T. de Ruiter. 2016. Duality in two-stage adaptive linear optimization: Faster computation and stronger bounds. *INFORMS Journal on Computing* **28**(3) 500–511.

Bertsimas, D., X. V. Doan, K. Natarajan, C.-P. Teo. 2010. Models for minimax stochastic linear optimization problems with risk aversion. *Mathematics of Operations Research* **35**(3) 580–602.

Bertsimas, D., V. Goyal. 2012. On the power and limitations of affine policies in two-stage adaptive optimization. *Mathematical Programming* **134**(2) 491–531.

Bertsimas, D., D. A. Iancu, P. A. Parrilo. 2011. A hierarchy of near-optimal policies for multi-stage adaptive optimization. *IEEE Transactions on Automatic Control* **56**(12) 2809–2824.

Bertsimas, D., E. Litvinov, X. A. Sun, J. Zhao, T. Zheng. 2013. Adaptive robust optimization for the security constrained unit commitment problem. *IEEE Transactions on Power Systems* **28**(1) 52–63.

Bertsimas, D., M. Sim. 2004. The price of robustness. *Operations Research* **52**(1) 35–53.

Bleichrodt, H., A. Cillo, E. Diecidue. 2010. A quantitative measurement of regret theory. *Management Science* **56**(1) 161–175.

Borodin, A., R. El-Yaniv. 2005. *Online computation and competitive analysis*. Cambridge university press.

Caldentey, R., Y. Liu, I. Lobel. 2017. Intertemporal pricing under minimax regret. *Operations Research* **65**(1) 104–129.

Chen, B., J. Wang, L. Wang, Y. He, Z. Wang. 2014. Robust optimization for transmission expansion planning: Minimax cost vs. minimax regret. *IEEE Transactions on Power Systems* **29**(6) 3069–3077.

Chen, L., S. He, S. Zhang. 2011. Tight bounds for some risk measures, with applications to robust portfolio selection. *Operations Research* **59**(4) 847–865.

Chen, X., M. Sim, P. Sun, J. Zhang. 2008. A linear decision-based approximation approach to stochastic programming. *Operations Research* **56**(2) 344–357.

Chen, X., Y. Zhang. 2009. Uncertain linear programs: Extended affinely adjustable robust counterparts. *Operations Research* **57**(6) 1469–1482.

Choi, B.-C., K. Chung. 2016. Min-max regret version of a scheduling problem with outsourcing decisions under processing time uncertainty. *European Journal of Operational Research* **252**(2) 367 – 375.

Conde, E. 2005. On the complexity of the continuous unbounded knapsack problem with uncertain coefficients. *Operations Research Letters* **33**(5) 481 – 485.

Conde, E. 2007. Minmax regret location-allocation problem on a network under uncertainty. *European Journal of Operational Research* **179**(3) 1025 – 1039.

Delage, E., D. A. Iancu. 2015. Robust multistage decision making. *INFORMS TutORials in Operations Research* 20–46.

Dunning, I., J. Huchette, M. Lubin. 2017. JuMP: A modeling language for mathematical optimization. *SIAM Review* **59**(2) 295–320.

Furini, F., M. Iori, S. Martello, M. Yagiura. 2015. Heuristic and exact algorithms for the interval min-max regret knapsack problem. *INFORMS Journal on Computing* **27**(2) 392–405.

Gabrel, V., C. Murat. 2010. Robustness and duality in linear programming. *Journal of the Operational Research Society* **61**(8) 1288–1296.

Gauvin, C., E. Delage, M. Gendreau. 2017. Decision rule approximations for the risk averse reservoir management problem. *European Journal of Operational Research* **261**(1) 317 – 336.

Goh, J., M. Sim. 2011. Robust optimization made easy with rome. *Operations Research* **59**(4) 973–985.

Guslitser, E. 2002. Uncertainty-immunized solutions in linear programming. Master's thesis, Technion, Israeli Institute of Technology.

Hadjiyiannis, M. J., P. J. Goulart, D. Kuhn. 2011. A scenario approach for estimating the suboptimality of linear decision rules in two-stage robust optimization. *2011 50th IEEE Conference on Decision and Control and European Control Conference*. 7386–7391.

Hanasusanto, G. A., D. Kuhn. 2018. Conic programming reformulations of two-stage distributionally robust linear programs over wasserstein balls. *Operations Research* **66**(3) 849–869.

Inuiguchi, M., H. Higashitani, T. Tanino. 1999. On computation methods of the minimax regret solution for linear programming problems with uncertain objective function coefficients. *IEEE SMC'99 Conference Proceedings. 1999 IEEE International Conference on Systems, Man, and Cybernetics (Cat. No.99CH37028)*, vol. 3. 979–984 vol.3.

Inuiguchi, M., M. Sakawa. 1995. Minimax regret solution to linear programming problems with an interval objective function. *European Journal of Operational Research* **86**(3) 526 – 536.

Inuiguchi, M., M. Sakawa. 1996. Maximum regret analysis in linear programs with an interval objective function. *Proceedings of International Workshop on Soft Computing in Industry '96* 308 – 317.

Inuiguchi, M., M. Sakawa. 1997a. An achievement rate approach to linear programming problems with an interval objective function. *The Journal of the Operational Research Society* **48**(1) 25–33.

Inuiguchi, M., M. Sakawa. 1997b. Minimax regret solution algorithms for linear program with convex polyhedral objective coefficients. *Proceedings of Second European Workshop on Fuzzy Decision Analysis and Neural Networks for Management, Planning and Optimization, Dortmund, Germany* 116–125.

Inuiguchi, M., T. Tanino. 2001. On computation methods for a minimax regret solution based on outer approximation and cutting hyperplanes. *International Joural of Fuzzy Systems* **3**(4) 548–557.

Inuiguchi, Masahiro, Yasufumi Kume. 1994. Minimax regret in linear programming problems with an interval objective function. G. H. Tzeng, H. F. Wang, U. P. Wen, P. L. Yu, eds., *Multiple Criteria Decision Making*. Springer New York, New York, NY, 65–74.

Jiang, R., J. Wang, M. Zhang, Y. Guan. 2013. Two-stage minimax regret robust unit commitment. *IEEE Transactions on Power Systems* **28**(3) 2271–2282.

Kasperski, A., A. Kurpisz, P. Zieliński. 2012. Approximating a two-machine flow shop scheduling under discrete scenario uncertainty. *European Journal of Operational Research* **217**(1) 36 – 43.

Kim, B. S., B. D. Chung. 2017. Affinely adjustable robust model for multiperiod production planning under uncertainty. *IEEE Trans. Eng. Manag.* **64**(4) 505–514.

Kouvelis, P., G. Yu. 1996. *Robust Discrete Optimization and Its Applications*. Nonconvex Optimization and Its Applications, Springer US.

Kuhn, D., W. Wiesemann, A. Georghiou. 2011. Primal and dual linear decision rules in stochastic and robust optimization. *Mathematical Programming* **130**(1) 177–209.

Lin, J., T. S. Ng. 2011. Robust multi-market newsvendor models with interval demand data. *European Journal of Operational Research* **212**(2) 361–373.

Loomes, G., R. Sugden. 1982. Regret theory: An alternative theory of rational choice under uncertainty. *The economic journal* **92**(368) 805–824.

Mausser, H. E., M. Laguna. 1998. A new mixed integer formulation for the maximum regret problem. *International Transactions in Operational Research* **5**(5) 389 – 403.

Mausser, H. E., M. Laguna. 1999a. A heuristic to minimax absolute regret for linear programs with interval objective function coefficients. *European Journal of Operational Research* **117**(1) 157 – 174.

Mausser, H. E., M. Laguna. 1999b. Minimising the maximum relative regret for linear programmes with interval objective function coefficients. *The Journal of the Operational Research Society* **50**(10) 1063–1070.

Melamed, M., A. Ben-Tal, B. Golany. 2016. On the average performance of the adjustable RO and its use as an offline tool for multi-period production planning under uncertainty. *Computational Management Science* **13**(2) 1–23.

Milnor, John Willard. 1954. Games against nature. R. M. Thrall, C. H. Coombs, R. L. Davis, eds., *Decision Processes*. New York: Wiley.

Minoux, M. 2009. On robust maximum flow with polyhedral uncertainty sets. *Optimization lettersl* **3**(3) 367–376.

Natarajan, K., D. Shi, K.-C. Toh. 2014. A probabilistic model for minmax regret in combinatorial optimization. *Operations Research* **62**(1) 160–181.

Ng, K. K. H., C. K. M. Lee, F. T. S. Chan, Y. Qin. 2017. Robust aircraft sequencing and scheduling problem with arrival/departure delay using the min-max regret approach. *Transportation Research Part E: Logistics and Transportation Review* **106** 115 – 136.

Ng, T. S. 2013. Robust regret for uncertain linear programs with application to co-production models. *European Journal of Operational Research* **227**(3) 483 – 493.

Ning, C., F. You. 2018. Adaptive robust optimization with minimax regret criterion: Multi-objective optimization framework and computational algorithm for planning and scheduling under uncertainty. *Computers & Chemical Engineering* **108** 425 – 447.

Perakis, G., G. Roels. 2008. Regret in the newsvendor model with partial information. *Operations Research* **56**(1) 188–203.

Pérez-Galarce, F., A. Candia-Vjar, C. Astudillo, M. Bardeen. 2018. Algorithms for the minmax regret path problem with interval data. *Information Sciences* **462** 218 – 241.

Pochet, Y., L. A. Wolsey. 1988. Lot-size models with backlogging: Strong reformulations and cutting planes. *Mathematical Programming* **40**(1-3) 317–335.

Savage, L. J. 1951. The theory of statistical decision. *Journal of the American Statistical association* **46**(253) 55–67.

Simchi-Levi, D., N. Trichakis, P. Y. Zhang. 2016. Designing response supply chain against bioattacks. *to appear in Operations Research* .

Stoye, J. 2011. Statistical decisions under ambiguity. *Theory and decision* **70**(2) 129–148.

Vairaktarakis, G. L. 2000. Robust multi-item newsboy models with a budget constraint. *International Journal of Production Economics* **66**(3) 213–226.

Xu, G., S. Burer. 2018. A copositive approach for two-stage adjustable robust optimization with uncertain right-hand sides. *Computational Optimization and Applications* **70**(1) 33–59.

Yanikoglu, I., B. L. Gorissen, D. den Hertog. 2018. A survey of adjustable robust optimization. *European Journal of Operational Research* .

Yue, J., B. Chen, M.-C. Wang. 2006. Expected value of distribution information for the newsvendor problem. *Operations Research* **54**(6) 1128–1136.

Z., Muhong. 2011. Two-stage minimax regret robust uncapacitated lot-sizing problems with demand uncertainty. *Operations Research Letters* **39**(5) 342 – 345.

Zeng, B., L. Zhao. 2013. Solving two-stage robust optimization problems using a column-and-constraint generation method. *Operations Research Letters* **41**(5) 457 – 461.

Zhen, J., D. den Hertog, M. Sim. 2018. Adjustable robust optimization via fourier–motzkin elimination. *Operations Research* **66**(4) 1086–1100.

Zhu, Z., J. Zhang, Y. Ye. 2013. Newsvendor optimization with limited distribution information. *Optimization Methods & Software* **28**(3) 640 – 667.

A Ning and You (2018)'s C&CG Approach is an Optimistic Approximation

Consider the multi-item newsvendor problem presented in section 7.1 where we let $n_x = n_y = n_\zeta = 2$ items, the sale price be $p_i = 1$, ordering cost $c_i = 1$, salvage value $s_i = 0$, and shortage cost $b_i = 1$. We also consider that the two items have a nominal value of 50 and 25 with maximum deviation of 50 and 25 respectively and that the sum of absolute relative deviations must be smaller or equal to one, i.e. $\Gamma = 1$. Moreover, we consider that the maximum total number of items ordered must be smaller than 100, namely that $\mathcal{X} := \{\boldsymbol{x} \in \mathbb{R}^2_+ \,|\, x_1 + x_2 \leq 100\}$. In this context, one can show numerically that the minimal worst-case absolute regret is equal to 45.833 and achieved by ordering 44.657 units of item #1 and 23.824 units of item #2. On the other hand, the C&CG approach proposed in Ning and You (2018) recommends ordering 37.5 units of item #1 and 25 units of item #2, estimating the minimal worst-case absolute regret achieved by this solution to be 37.5 when it is actually of 54.167. In particular, when the solution (37.5, 25) is used, one can easily confirm that if only the integer values of δ^+ and δ^- are considered in the uncertainty set, then for all possible cases the regret achieved is 37.5. However, this is an underestimation of the regret that is achieved over $\mathcal{U}(\Gamma)$ since at $\zeta = (250/3,\, 50/3)$ is equal to $325/6 \approx 54.167$. This confirms that C&CG approach proposed in Ning and You (2018) solves an optimistic approximation of the WCARM problem.

B TSLRO Reformulations for WCRRM in cost minimization problems

Given a non-negative optimal second-stage cost function $f(\boldsymbol{x}, \boldsymbol{\zeta})$ which depends on both the decision and the realization of some uncertain vector of parameters $\boldsymbol{\zeta}$, following the formulation presented in Mausser and Laguna (1999b), one measures the relative regret experienced once $\boldsymbol{\zeta}$ is revealed as the ratio of the difference between the lowest cost achievable $\min_{\boldsymbol{x}' \in \mathcal{X}} f(\boldsymbol{x}', \boldsymbol{\zeta})$ and the cost $f(\boldsymbol{x}, \boldsymbol{\zeta})$ achieved by the decision $\boldsymbol{x}$ that was implemented, over the lowest cost achievable. The worst-case relative regret minimization (WCRRM) problem thus takes the form:

$$\underset{\boldsymbol{x} \in \mathcal{X}}{\text{minimize}} \; \sup_{\boldsymbol{\zeta} \in \mathcal{U}} \left\{ \frac{f(\boldsymbol{x}, \boldsymbol{\zeta}) - \inf_{\boldsymbol{x}' \in \mathcal{X}} f(\boldsymbol{x}', \boldsymbol{\zeta})}{\inf_{\boldsymbol{x}' \in \mathcal{X}} f(\boldsymbol{x}', \boldsymbol{\zeta})} \right\}, \tag{49}$$

where, when $\inf_{\boldsymbol{x}' \in \mathcal{X}} f(\boldsymbol{x}', \boldsymbol{\zeta}) = 0$, we should interpret the relative regret as being either 0 if $f(\boldsymbol{x}, \boldsymbol{\zeta}) = 0$ or infinite otherwise. Equivalently, in terms of $h(\boldsymbol{x}, \boldsymbol{\zeta}) := -f(\boldsymbol{x}, \boldsymbol{\zeta})$, we will define the WCRRM problem has:

$$\text{(WCRRM)} \quad \underset{\boldsymbol{x} \in \mathcal{X}}{\text{minimize}} \; \sup_{\boldsymbol{\zeta} \in \mathcal{U}} \left\{ \frac{h(\boldsymbol{x}, \boldsymbol{\zeta}) - \sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})}{\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})} \right\}. \tag{50}$$

As mentioned above, we make the following assumption about the profit function in this two-stage problem.

Assumption 8. *The cost function $h(\boldsymbol{x}, \boldsymbol{\zeta}) \leq 0$ for all $\boldsymbol{x} \in \mathcal{X}$ and all $\boldsymbol{\zeta} \in \mathcal{U}$. This implies that assumptions 3 and 5 are satisfied and that the optimal value of problems* (50) *is greater equal zero.*

In what follows we demonstrate how the WCRRM problem can be reformulated as a TSLRO when the cost function $f(\boldsymbol{x}, \boldsymbol{\zeta})$ (a.k.a. $-h(\mathcal{X}, \boldsymbol{\zeta})$) captures the cost of a second-stage linear decision model with either right-hand side or objective uncertainty.

B.1 The Case of Right-Hand Side Uncertainty

We consider the case where $h(\boldsymbol{x}, \boldsymbol{\zeta})$ takes the form presented in problem (3) and where uncertainty is limited to the right-hand side as defined in Definition 1.

Proposition 11. *Given that assumptions 1 and 8 are satisfied, the cost-based WCRRM problem with right-hand side uncertainty is equivalent to the following TSLRO problem:*

$$\underset{\boldsymbol{x}' \in \mathcal{X}', \boldsymbol{y}'(\cdot)}{\text{maximize}} \quad \inf_{\boldsymbol{\zeta}' \in \mathcal{U}'} \boldsymbol{c}'^T \boldsymbol{x}' \tag{51a}$$

$$\text{subject to} \quad A'\boldsymbol{x}' + B'\boldsymbol{y}'(\boldsymbol{\zeta}') \leq \Psi'(\boldsymbol{x}')\boldsymbol{\zeta}' + \boldsymbol{\psi}', \, \forall \boldsymbol{\zeta}' \in \mathcal{U}' \tag{51b}$$

where $x' \in \mathbb{R}^{n_x+1}$, $\boldsymbol{\zeta}' \in \mathbb{R}^{n_\zeta+n_x+n_y}$, $\boldsymbol{y}' : \mathbb{R}^{n_\zeta+n_x+n_y} \to \mathbb{R}^{n_y}$, $\boldsymbol{c}' = [-1 \quad \boldsymbol{0}^T]^T$, *while* $\mathcal{X}' := \{[t \;\; \boldsymbol{x}^T]^T \in \mathbb{R}^{n_x+1} \,|\, \boldsymbol{x} \in \mathcal{X}, \, t \geq 0\}$, $\mathcal{U}'$ *is defined as in equation* (17) *and*

$$A' = \begin{bmatrix} 0 & -\boldsymbol{c}^T \\ 0 & A \end{bmatrix}, \quad B' = \begin{bmatrix} -\boldsymbol{d}^T \\ B \end{bmatrix}, \quad \Psi'(\boldsymbol{x}') = \begin{bmatrix} \boldsymbol{0}^T & -\boldsymbol{c}^T & -\boldsymbol{d}^T \\ \Psi & 0 & 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{0}^T & -\boldsymbol{c}^T & -\boldsymbol{d}^T \\ 0 & 0 & 0 \end{bmatrix} x_1', \quad \boldsymbol{\psi}' := \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\psi} \end{bmatrix}.$$

In particular, a solution for the WCRRM takes the form of $\boldsymbol{x}^* := \boldsymbol{x}'^*_{2:n_x+1}$ *and achieves a worst-case relative regret of* x_1'. *Furthermore, this TSLRO reformulation necessarily satisfies Assumption 1 while it only satisfies Assumption 2 if all* $\boldsymbol{x} \in \mathcal{X}$ *achieve a worst-case regret of zero.*

Proof. We first employ an epigraph form for problem (50) as follows:

$$\underset{\boldsymbol{x} \in \mathcal{X}, t}{\text{minimize}} \quad t \tag{52a}$$

$$\text{subject to} \quad \sup_{\boldsymbol{\zeta} \in \mathcal{U}} \left\{ \frac{h(\boldsymbol{x}, \boldsymbol{\zeta}) - \sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})}{\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})} \right\} \leq t \tag{52b}$$

$$0 \leq t \tag{52c}$$

where we impose that $0 \leq t$ since Assumption 8 ensures that the optimal value of the WCRRM problem is greater equal zero. One can then manipulate constraint (52b) to show it is equivalent to

$$\frac{h(\boldsymbol{x}, \boldsymbol{\zeta}) - \sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})}{\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})} \leq t \, , \, \forall \boldsymbol{\zeta} \in \mathcal{U}$$

hence to

$$h(\boldsymbol{x}, \boldsymbol{\zeta}) - \sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) \geq t(\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})) \, , \, \forall \boldsymbol{\zeta} \in \mathcal{U} \, ,$$

since, for a fixed $\boldsymbol{\zeta}$, either $\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}, \boldsymbol{\zeta}) < 0$ or otherwise the new constraint becomes equivalent to $h(\boldsymbol{x}, \boldsymbol{\zeta}) = 0$ which captures exactly the fact that the regret is zero under this $\boldsymbol{\zeta}$ scenario if $h(\boldsymbol{x}, \boldsymbol{\zeta}) = 0$ and otherwise infinite. Finally, we obtain the constraint:

$$(t+1) \sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) - h(\boldsymbol{x}, \boldsymbol{\zeta}) \leq 0 \, , \, \forall \boldsymbol{\zeta} \in \mathcal{U} \, . \tag{53}$$

By substituting problem (3) in this constraint, we obtain the following reformulations

$$\begin{aligned} (52\text{b}) \; &\equiv \; (t+1) \sup_{\boldsymbol{x}' \in \mathcal{X}, \boldsymbol{y}' \in \mathcal{Y}(\boldsymbol{x}', \boldsymbol{\zeta})} \boldsymbol{c}^T \boldsymbol{x}' + \boldsymbol{d}^T \boldsymbol{y}' - \sup_{\boldsymbol{y} \in \mathcal{Y}(\boldsymbol{x}, \boldsymbol{\zeta})} \boldsymbol{c}^T \boldsymbol{x} + \boldsymbol{d}^T \boldsymbol{y} \leq 0 \, , \forall \boldsymbol{\zeta} \in \mathcal{U} \\ &\equiv \; \min_{\boldsymbol{y} \in \mathcal{Y}(\boldsymbol{x}, \boldsymbol{\zeta})} -\boldsymbol{c}^T \boldsymbol{x} - \boldsymbol{d}^T \boldsymbol{y} + (1+t)\boldsymbol{c}^T \boldsymbol{x}' + (1+t)\boldsymbol{d}^T \boldsymbol{y}' \leq \boldsymbol{0} \, , \forall \boldsymbol{\zeta} \in \mathcal{U}, x' \in \mathcal{X}, \boldsymbol{y}' \in \mathcal{Y}(\boldsymbol{x}', \boldsymbol{\zeta}) \, . \end{aligned}$$

Hence the WCRRM problem reduces to:

$$\underset{\boldsymbol{x} \in \mathcal{X}, t \geq 0}{\text{minimize}} \quad \sup_{\boldsymbol{\zeta} \in \mathcal{U}, \boldsymbol{x}' \in \mathcal{X}, \boldsymbol{y}' \in \mathcal{Y}(\boldsymbol{x}', \boldsymbol{\zeta})} h'(\boldsymbol{x}, t, \boldsymbol{\zeta}, \boldsymbol{x}', \boldsymbol{y}')$$

where

$$h'(\boldsymbol{x}, t, \boldsymbol{\zeta}, \boldsymbol{x}', \boldsymbol{y}') := \begin{array}{ll} \inf_{\boldsymbol{y}} & t \\ \text{s.t.} & -\boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{d}^T\boldsymbol{y} \leq -(t+1)\boldsymbol{c}^T\boldsymbol{x}' - (t+1)\boldsymbol{d}^T\boldsymbol{y}' \\ & A\boldsymbol{x} + B\boldsymbol{y} \leq \Psi\boldsymbol{\zeta} + \boldsymbol{\psi} \end{array}$$

This problem can be rewritten in the form presented in equation (51).

Note that the arguments to support the conditions under which assumptions 1 and 2 are satisfies are exactly the same as in the proof of Proposition 5. □

B.2 The Case of Objective Uncertainty

We consider the case where $h(\boldsymbol{x}, \boldsymbol{\zeta})$ takes the form presented in problem (3).

Proposition 12. *Given that assumptions 1 and 8 are satisfied, the WCRRM problem with objective uncertainty is equivalent to the following TSLRO problem:*

$$\begin{aligned} &\underset{\boldsymbol{x}', \boldsymbol{y}'(.)}{\text{maximize}} && \inf_{\boldsymbol{\zeta}' \in \mathcal{U}'} \boldsymbol{c}'^T\boldsymbol{x}' && (54a) \\ &\text{subject to} && A'\boldsymbol{x}' + B'\boldsymbol{y}'(\boldsymbol{\zeta}') \leq \Psi'(\boldsymbol{x}')\boldsymbol{\zeta}' + \boldsymbol{\psi}' && (54b) \\ &&& \boldsymbol{x}' \in \mathcal{X}' && (54c) \end{aligned}$$

where $\boldsymbol{x}' \in \mathbb{R}^{n_x+1}$, $\boldsymbol{y}' : \mathbb{R}^{n_\zeta+m} \to \mathbb{R}^{m+r}$, *while* $\mathcal{X}' := \{[u \;\; \boldsymbol{z}^T]^T \in \mathbb{R}^{n_x+1} \,|\, W\boldsymbol{z} \geq \boldsymbol{v}u, \, -1 \leq u \leq 0\}$, $\mathcal{U}'$ *is defined as in equation* (21)*. Moreover, we have that* $\boldsymbol{c}' := [-1 \;\; \boldsymbol{0}^T]^T$, *while*

$$A' := \begin{bmatrix} 0 & \boldsymbol{c}^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B' := \begin{bmatrix} \boldsymbol{\psi}^T & \boldsymbol{v}^T \\ A^T & W^T \\ -A^T & -W^T \\ B^T & 0 \\ -B^T & 0 \\ -I & 0 \\ 0 & -I \end{bmatrix}, \quad \Psi'(\boldsymbol{x}') := \begin{bmatrix} \boldsymbol{0}^T & -\boldsymbol{\psi}^T\boldsymbol{x}'_1 + \boldsymbol{x}'^T_{2:n_x+1}A^T \\ 0 & 0 \\ 0 & 0 \\ D & 0 \\ -D & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{\psi}' := \begin{bmatrix} 0 \\ \boldsymbol{c} \\ -\boldsymbol{c} \\ \boldsymbol{d} \\ -\boldsymbol{d} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}.$$

In particular, a solution for the WCRRM takes the form of $\boldsymbol{x}^* := \boldsymbol{x}'^*_{2:n_x+1}/x'^*_1$ *and achieves a worst-case relative regret of* $-1 - 1/x'^*_1$ *if* $x'^*_1 < 0$ *while the best worst-case relative regret should be considered infinite if* $x'^*_1 = 0$*. Furthermore, this TSLRO reformulation necessarily satisfies Assumption 1 while it only satisfies Assumption 2 if all* $\boldsymbol{x} \in \mathcal{X}$ *achieve a worst-case regret of zero.*

Proof. The first steps of this proof are exactly as in the proof of Proposition 11 up to equation (53). The next steps are then exactly analogous to the steps followed in the proof of Proposition 6 which we repeat for completeness. Since we are now dealing with objective uncertainty, we substitute $h(\boldsymbol{x}, \boldsymbol{\zeta})$ and $\sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta})$ using their respective dual form (see equations (13) and (23) respectively). Strong duality applies since Assumption 8 implies assumptions 3 and 5 are satisfied, which results to the following reformulation:

$$\begin{aligned} (52b) \;\equiv\; & (t+1) \sup_{\boldsymbol{x}' \in \mathcal{X}} h(\boldsymbol{x}', \boldsymbol{\zeta}) - h(\boldsymbol{x}, \boldsymbol{\zeta}) \leq 0 \,, \forall \boldsymbol{\zeta} \in \mathcal{U} && (55) \\ \equiv\; & (t+1) \inf_{(\boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \Upsilon_1(\boldsymbol{\zeta})} \boldsymbol{\psi}^T\boldsymbol{\lambda} + \boldsymbol{v}^T\boldsymbol{\gamma} - \inf_{\boldsymbol{\rho} \in \Upsilon_2(\boldsymbol{\zeta})} \{\boldsymbol{c}^T\boldsymbol{x} + (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho}\} \leq 0, \forall \boldsymbol{\zeta} \in \mathcal{U} && (56) \\ \equiv\; & \inf_{(\boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \Upsilon_1(\boldsymbol{\zeta})} (1+t)\boldsymbol{\psi}^T\boldsymbol{\lambda} + (1+t)\boldsymbol{v}^T\boldsymbol{\gamma} - \boldsymbol{c}^T\boldsymbol{x} - (\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \leq 0, \forall \boldsymbol{\zeta} \in \mathcal{U}, \boldsymbol{\rho} \in \Upsilon_2(\boldsymbol{\zeta}) && (57) \\ \equiv\; & \inf_{(\boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \Upsilon_1(\boldsymbol{\zeta})} \boldsymbol{\psi}^T\boldsymbol{\lambda} + \boldsymbol{v}^T\boldsymbol{\gamma} - \frac{1}{1+t}\boldsymbol{c}^T\boldsymbol{x} - \frac{1}{1+t}(\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \leq 0, \forall \boldsymbol{\zeta} \in \mathcal{U}, \boldsymbol{\rho} \in \Upsilon_2(\boldsymbol{\zeta}) && (58) \end{aligned}$$

where $\Upsilon_1(\boldsymbol{\zeta})$ and $\Upsilon_2(\boldsymbol{\zeta})$ are as defined in the proof of Proposition 4. Hence the WCRRM problem reduces to:

$$\underset{\boldsymbol{x}\in\mathcal{X},t\geq 0}{\text{minimize}} \quad \sup_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} h''(\boldsymbol{x},t,\boldsymbol{\zeta},\boldsymbol{\rho})$$

where

$$\begin{aligned} h''(\boldsymbol{x},t,\boldsymbol{\zeta},\boldsymbol{\rho}) := \quad & \inf_{\boldsymbol{\lambda},\boldsymbol{\gamma}} && t \\ & \text{s.t.} && \boldsymbol{\psi}^T\boldsymbol{\lambda} + \boldsymbol{v}^T\boldsymbol{\gamma} - \frac{1}{1+t}\boldsymbol{c}^T\boldsymbol{x} - \frac{1}{1+t}(\boldsymbol{\psi} - A\boldsymbol{x})^T\boldsymbol{\rho} \leq 0 \\ &&& A^T\boldsymbol{\lambda} + W^T\boldsymbol{\gamma} = \boldsymbol{c} \\ &&& B^T\boldsymbol{\lambda} = \boldsymbol{d}(\boldsymbol{\zeta}) \\ &&& \boldsymbol{\lambda} \geq 0,\ \boldsymbol{\gamma} \geq 0\,. \end{aligned}$$

Using a simple replacement of variables $u := -\frac{1}{1+t}$ and $\boldsymbol{z} := -\frac{1}{1+t}\boldsymbol{x}$ and applying a monotone transformation of the objective function $t \to -\frac{1}{1+t}$, we obtain that the WCRRM is equivalently represented as

$$\underset{-1\leq u<0,\,\boldsymbol{z}:W\boldsymbol{z}\geq \boldsymbol{v}u}{\text{minimize}} \quad \sup_{\boldsymbol{\zeta}\in\mathcal{U},\boldsymbol{\rho}\in\Upsilon_2(\boldsymbol{\zeta})} h''(\boldsymbol{z},u,\boldsymbol{\zeta},\boldsymbol{\rho})$$

where

$$\begin{aligned} h''(\boldsymbol{z},u,\boldsymbol{\zeta},\boldsymbol{\rho}) := \quad & \inf_{\boldsymbol{\lambda},\boldsymbol{\gamma}} && u \\ & \text{s.t.} && \boldsymbol{\psi}^T\boldsymbol{\lambda} + \boldsymbol{v}^T\boldsymbol{\gamma} + \boldsymbol{c}^T\boldsymbol{z} + (u\boldsymbol{\psi} - A\boldsymbol{z})^T\boldsymbol{\rho} \leq 0 \\ &&& A^T\boldsymbol{\lambda} + W^T\boldsymbol{\gamma} = \boldsymbol{c} \\ &&& B^T\boldsymbol{\lambda} = \boldsymbol{d}(\boldsymbol{\zeta}) \\ &&& \boldsymbol{\lambda} \geq 0,\ \boldsymbol{\gamma} \geq 0\,. \end{aligned}$$

This problem can be rewritten in the form presented in equation (54) as long as when the optimal value of the TSLRO is 0 one concludes that best worst-case relative regret is infinite.

Note that the arguments to support the conditions under which assumptions 1 and 2 are satisfies are similar as in the proof of Proposition 6. □

C Main step of proof of Proposition 10

The main step of the proof of Proposition 10 consists in establishing the following lemma.

Lemma 13. *If $h(\boldsymbol{x},\boldsymbol{\zeta})$ is a sum of of piecewise linear concave functions of the form presented in* (44)*, the uncertainty set $\mathcal{U}$ is the budgeted uncertainty set defined as in* (45)*, and either of the following conditions are satisfied:*

i. $\Gamma = 1$

ii. $\Gamma = n_\zeta$ *and uncertainty is "additive": i.e.* $\boldsymbol{\alpha}_{ik}(\boldsymbol{x}) = \bar{\alpha}_{ik}(\boldsymbol{x})(\sum_{\ell<i}\hat{\alpha}_\ell(\boldsymbol{x})\boldsymbol{e}_\ell)$ *for some* $\bar{\alpha}_{ik} : \mathbb{R}^{n_x} \to \mathbb{R}$ *for all i and k and some* $\hat{\boldsymbol{\alpha}} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_\zeta}$

iii. Γ *is integer and objective is "decomposable": i.e.* $\boldsymbol{\alpha}_{ik}(\boldsymbol{x}) = \bar{\alpha}_{ik}(\boldsymbol{x})\boldsymbol{e}_i$ *for some* $\bar{\alpha}_{ik} : \mathbb{R}^{n_x} \to \mathbb{R}$ *for all i and k*

then, affine decision rules with respect to $(\boldsymbol{\delta}^+,\boldsymbol{\delta}^-)$ are optimal in the following two-stage linear programming formulation of $\text{maximize}_{\boldsymbol{x}\in\mathcal{X}}\ \min_{\boldsymbol{\zeta}\in\mathcal{U}}\ h(\boldsymbol{x},\boldsymbol{\zeta}) - \bar{\gamma} - \boldsymbol{\gamma}^T\boldsymbol{\zeta}$*:*

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathcal{X},\boldsymbol{y}(\cdot,\cdot)}{\text{maximize}} \quad & \min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} \sum_{i=1}^{n_y} y_i(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) - \bar{\gamma} - \boldsymbol{\gamma}^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) \\ \text{subject to} \quad & y_i(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \leq \boldsymbol{\alpha}_{ik}(\boldsymbol{x})^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + \beta_{ik}(\boldsymbol{x})\,,\ \forall\,(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \in \mathcal{U}_\pm(\Gamma),\ \forall\, i,\ \forall\, k\,, \end{aligned}$$

where $\boldsymbol{y} : \mathbb{R}^{n_\zeta} \times \mathbb{R}^{n_\zeta} \to \mathbb{R}^{n_y}$.

Proof. For each of the three cases, we will demonstrate that there exists a linear transformation of $\boldsymbol{y}(\cdot)$ that can be used to distribute the term $\bar{\gamma}+\boldsymbol{\gamma}^T(\boldsymbol{\zeta}^+-\boldsymbol{\zeta}^-)$ in the constraints while preserving their respective structure. This then allows us to exploit Corollary 1 in Ardestani-Jaafari and Delage (2016) to reach our conclusion.

Conditions i.: Let us start by characterizing for any fixed $\boldsymbol{x} \in \mathcal{X}$, the optimal value of the adversarial problem as $h_1(\boldsymbol{x})$, namely:

$$h_1(\boldsymbol{x}) := \min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} h(\boldsymbol{x}, \boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) - \bar{\gamma} - \boldsymbol{\gamma}^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-)$$

and by $h_2(\boldsymbol{x})$ the lower bound on this value obtained using affine decision rules:

$$h_2(\boldsymbol{x}) := \max_{\bar{\boldsymbol{y}},\{\boldsymbol{y}_i^+,\boldsymbol{y}_i^-\}_{i=1}^{n_y}} \min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} \sum_{i=1}^{n_y} (\bar{y}_i + {\boldsymbol{y}_i^+}^T\boldsymbol{\zeta}^+ + {\boldsymbol{y}_i^-}^T\boldsymbol{\zeta}^-) - \bar{\gamma} - \boldsymbol{\gamma}^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) \tag{59a}$$

$$\text{s.t.} \quad \bar{y}_i + {\boldsymbol{y}_i^+}^T\boldsymbol{\zeta}^+ + {\boldsymbol{y}_i^-}^T\boldsymbol{\zeta}^- \le \boldsymbol{\alpha}_{ik}(\boldsymbol{x})^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + \beta_{ik}(\boldsymbol{x}), \quad \forall (\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \in \mathcal{U}_\pm(\Gamma) \; \forall i, \forall k. \tag{59b}$$

We will show that $h_2(\boldsymbol{x})$ is actually equal to $h_1(\boldsymbol{x})$. In particular, by replacing $\bar{z}_1 := \bar{y}_1 - \bar{\gamma}$, $\boldsymbol{z}_1^+ := \boldsymbol{y}_1^+ - \boldsymbol{\gamma}$, $\boldsymbol{z}_1^- := \boldsymbol{y}_1^- + \boldsymbol{\gamma}$, while $\bar{z}_i = \bar{y}_i$, $\boldsymbol{z}_i^+ : \boldsymbol{y}_i^+$, and $\boldsymbol{z}_i^- := \boldsymbol{y}_i^-$ for all $i \ge 2$, we then get that:

$$\begin{aligned}
h_2(\boldsymbol{x}) := &\max_{\bar{\boldsymbol{z}},\{\boldsymbol{z}_i^+,\boldsymbol{z}_i^-\}_{i=1}^{n_y}} \min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} \sum_{i=1}^{n_y} (\bar{z}_i + {\boldsymbol{z}_i^+}^T\boldsymbol{\zeta}^+ + {\boldsymbol{z}_i^-}^T\boldsymbol{\zeta}^-) \\
&\text{s.t.} \\
&\bar{z}_1 + {\boldsymbol{z}_1^+}^T\boldsymbol{\zeta}^+ + {\boldsymbol{z}_1^-}^T\boldsymbol{\zeta}^- \le (\boldsymbol{\alpha}_{1k}(\boldsymbol{x}) - \boldsymbol{\gamma})^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + (\beta_{1k}(\boldsymbol{x}) - \bar{\gamma}), \forall (\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \in \mathcal{U}_\pm(\Gamma), \forall k \\
&\bar{z}_i + {\boldsymbol{z}_i^+}^T\boldsymbol{\zeta}^+ + {\boldsymbol{z}_i^-}^T\boldsymbol{\zeta}^- \le \boldsymbol{\alpha}_{ik}(\boldsymbol{x})^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + \beta_{ik}(\boldsymbol{x}), \forall (\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \in \mathcal{U}_\pm(\Gamma), \forall i \ge 2, \forall k.
\end{aligned}$$

One can easily recognize that this form is equivalent to the lower bound obtained when applying affine decision rules to approximate the worst-case value of a sum of piecewise linear concave functions. Following Corollary 1 in Ardestani-Jaafari and Delage (2016), since $\Gamma = 1$, we can conclude that

$$\begin{aligned}
h_2(\boldsymbol{x}) := \max_{\boldsymbol{z}(\cdot)} \quad &\min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} \sum_{i=1}^{n_y} z_i(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \\
\text{s.t.} \quad &z_1(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \le (\boldsymbol{\alpha}_{1k}(\boldsymbol{x}) - \boldsymbol{\gamma})^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + (\beta_{1k}(\boldsymbol{x}) - \bar{\gamma}), \forall (\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \in \mathcal{U}_\pm(\Gamma), \forall k \\
&z_i(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \le \boldsymbol{\alpha}_{ik}(\boldsymbol{x})^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + \beta_{ik}(\boldsymbol{x}), \forall (\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \in \mathcal{U}_\pm(\Gamma), \forall i \ge 2, \forall k.
\end{aligned}$$

which once more with a replacement of variables gives us:

$$\begin{aligned}
h_2(\boldsymbol{x}) = \max_{\boldsymbol{y}(\cdot)} \quad &\min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} \sum_{i=1}^{n_y} y_i(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) - \bar{\gamma} - \boldsymbol{\gamma}^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) \\
\text{s.t.} \quad &z_i(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \le \boldsymbol{\alpha}_{ik}(\boldsymbol{x})^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + \beta_{ik}(\boldsymbol{x}) \, \forall (\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \in \mathcal{U}_\pm(\Gamma), \forall i, \forall k.
\end{aligned}$$

Hence, we have that $h_2(\boldsymbol{x}) = h_1(\boldsymbol{x})$.

Conditions iii.: The proof for Condition iii. is fairly similar except that we exploit a different affine transformation for passing from $\boldsymbol{y}$ to $\boldsymbol{z}$. In particular, now we can exploit the fact that the objective function in (59) can be equivalently written as:

$$\max_{\bar{\boldsymbol{y}},\{\boldsymbol{y}_i^+,\boldsymbol{y}_i^-\}_{i=1}^{n_y}} \min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} \sum_{i=1}^{n_y}(\bar{y}_i - \bar{\gamma}/n_y) + (\boldsymbol{y}_i^+ - \gamma_i\boldsymbol{e}_i)^T\boldsymbol{\zeta}^+ + (\boldsymbol{y}_i^- + \gamma_i\boldsymbol{e}_i)^T\boldsymbol{\zeta}^- .$$

We can now replace $\bar{\boldsymbol{z}} := \bar{\boldsymbol{y}} - \bar{\gamma}/n_y$ and each $\boldsymbol{z}_i^+ := \boldsymbol{y}_i^+ - \gamma_i\boldsymbol{e}_i$ and $\boldsymbol{z}_i^- := \boldsymbol{y}_i^- + \gamma_i\boldsymbol{e}_i$ to get:

$$\begin{aligned} h_2(\boldsymbol{x}) = &\max_{\bar{\boldsymbol{z}},\{\boldsymbol{z}_i^+,\boldsymbol{z}_i^-\}_{i=1}^{n_y}} \min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} \sum_{i=1}^{n_y}(\bar{z}_i + {\boldsymbol{z}_i^+}^T\boldsymbol{\zeta}^+ + {\boldsymbol{z}_i^-}^T\boldsymbol{\zeta}^-) \\ &\text{s.t.} \\ &\bar{z}_i + {\boldsymbol{z}_i^+}^T\boldsymbol{\zeta}^+ + {\boldsymbol{z}_i^-}^T\boldsymbol{\zeta}^- \leq (\boldsymbol{\alpha}_{ik}(\boldsymbol{x}) - \gamma_i\boldsymbol{e}_i)^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + \beta_{ik}(\boldsymbol{x}) - \bar{\gamma}/n_y\,,\,, \quad \begin{array}{l}\forall\,(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)\\ \forall\,i,\,\forall\,k\,.\end{array} \end{aligned}$$

One can again recognize that this form is equivalent to the lower bound obtained when applying affine decision rules to approximate the worst-case value of $h'(\boldsymbol{x},\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-)$ which is defined as the sum of piecewise linear concave functions using $\boldsymbol{\alpha}'_{ik}(\boldsymbol{x}) := \boldsymbol{\alpha}_{ik}(\boldsymbol{x}) - \gamma_i\boldsymbol{e}_i$ and $\beta'_{ik}(\boldsymbol{x}) := \beta_{ik}(\boldsymbol{x}) - \bar{\gamma}/n_y$. Following Corollary 1 in Ardestani-Jaafari and Delage (2016), we can conclude that

$$h_2(\boldsymbol{x}) = \min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} h'(\boldsymbol{x},\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-)$$

since by Condition iii. we have that:

$$\boldsymbol{\alpha}'_{ik}(\boldsymbol{x}) = \boldsymbol{\alpha}_{ik}(\boldsymbol{x}) - \gamma_i\boldsymbol{e}_i = \bar{\alpha}_{ik}(\boldsymbol{x})\boldsymbol{e}_i - \gamma_i\boldsymbol{e}_i = (\bar{\alpha}_{ik}(\boldsymbol{x}) - \gamma_i)\boldsymbol{e}_i\,,$$

hence Condition 3 in Ardestani-Jaafari and Delage (2016) is satisfied. We can therefore conclude that

$$\begin{aligned} h_2(\boldsymbol{x}) = \max_{\boldsymbol{z}(\cdot)} \quad &\min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} \sum_{i=1}^{n_y} z_i(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \\ \text{s.t.} \quad &z_i(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \leq (\boldsymbol{\alpha}_{ik}(\boldsymbol{x}) - \gamma_i\boldsymbol{e}_i)^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + \beta_{ik}(\boldsymbol{x}) - \bar{\gamma}/n_y\,,\,, \quad \begin{array}{l}\forall\,(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)\\ \forall\,i,\,\forall\,k\,,\end{array} \end{aligned}$$

which once more with a replacement of variable gives us:

$$\begin{aligned} h_2(\boldsymbol{x}) = \max_{\boldsymbol{y}(\cdot)} \quad &\min_{(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma)} \sum_{i=1}^{n_y} y_i(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) - \bar{\gamma} - \boldsymbol{\gamma}^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) \\ \text{s.t.} \quad &z_i(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-) \leq \boldsymbol{\alpha}_{ik}(\boldsymbol{x})^T(\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + \beta_{ik}(\boldsymbol{x})\,\forall\,(\boldsymbol{\zeta}^+,\boldsymbol{\zeta}^-)\in\mathcal{U}_\pm(\Gamma),\,\forall\,i,\,\forall\,k\,. \end{aligned}$$

Hence, we have that $h_2(\boldsymbol{x}) = h_1(\boldsymbol{x})$.

Condition ii.: The proof for Condition ii. is again entirely analogous with a new affine transformation for passing from $\boldsymbol{y}$ to $\boldsymbol{z}$. In particular, we first assume for simplicity of exposition that $\hat{\alpha}_\ell \neq 0$ for all $\ell = 1,\ldots,n_\zeta$ and that $n_y = n_\zeta + 1$. We then exploit the fact that:

$$\begin{aligned} \boldsymbol{\gamma} &= \sum_{\ell=1}^{n_\zeta}\gamma_\ell\boldsymbol{e}_\ell = \sum_{\ell=1}^{n_\zeta}\hat{\alpha}_\ell\boldsymbol{e}_\ell\left(\sum_{i=\ell}^{n_\zeta}\frac{\gamma_i}{\hat{\alpha}_i} - \sum_{i=\ell+1}^{n_\zeta}\frac{\gamma_i}{\hat{\alpha}_i}\right) = \sum_{i=1}^{n_\zeta-1}(\frac{\gamma_i}{\hat{\alpha}_i} - \frac{\gamma_{i+1}}{\hat{\alpha}_{i+1}})\sum_{\ell\leq i}\hat{\alpha}_\ell\boldsymbol{e}_\ell + \frac{\gamma_{n_\zeta}}{\hat{\alpha}_\zeta}\sum_{\ell=1}^{n_\zeta}\hat{\alpha}_\ell\boldsymbol{e}_\ell \\ &= \sum_{i=1}^{n_\zeta+1}\bar{\alpha}'_i(\sum_{\ell<i}\hat{\alpha}_\ell\boldsymbol{e}_\ell)\,, \end{aligned}$$

where

$$\bar{\alpha}'_i := \begin{cases} 0 & \text{if } i = 1 \\ \frac{\gamma_{i-1}}{\hat{\alpha}_{i-1}} - \frac{\gamma_i}{\hat{\alpha}_i} & \text{if } i \in \{2, \ldots, n_\zeta\} \\ \frac{\gamma_{n_\zeta}}{\hat{\alpha}_\zeta} & \text{if } i = n_\zeta + 1 \end{cases} .$$

We therefore have that the objective function in (59) can be reformulated as

$$\max_{\bar{\boldsymbol{y}}, \{\boldsymbol{y}_i^+, \boldsymbol{y}_i^-\}_{i=1}^{n_y}} \min_{(\boldsymbol{\zeta}^+, \boldsymbol{\zeta}^-) \in \mathcal{U}_\pm(\Gamma)} \sum_{i=1}^{n_y} (\bar{y}_i - \bar{\gamma}/n_y) + (\boldsymbol{y}_i^+ - \bar{\alpha}'_i(\sum_{\ell<i} \hat{\alpha}_\ell \boldsymbol{e}_\ell))^T \boldsymbol{\zeta}^+ + (\boldsymbol{y}_i^- + \bar{\alpha}'_i(\sum_{\ell<i} \hat{\alpha}_\ell \boldsymbol{e}_\ell))^T \boldsymbol{\zeta}^- .$$

By replacing $\bar{z}_i := \bar{y}_i - \bar{\gamma}/n_y$ as before, while $\boldsymbol{z}_i^+ := \boldsymbol{y}_i^+ - \bar{\alpha}'_i(\sum_{\ell<i} \hat{\alpha}_\ell \boldsymbol{e}_\ell)$ and $\boldsymbol{z}_i^- := \boldsymbol{y}_i^- + \bar{\alpha}'_i(\sum_{\ell<i} \hat{\alpha}_\ell \boldsymbol{e}_\ell))^T \boldsymbol{\zeta}^-$, we obtain:

$$\begin{aligned} h_2(\boldsymbol{x}) = & \max_{\bar{\boldsymbol{z}}, \{\boldsymbol{z}_i^+, \boldsymbol{z}_i^-\}_{i=1}^{n_y}} \min_{(\boldsymbol{\zeta}^+, \boldsymbol{\zeta}^-) \in \mathcal{U}_\pm(\Gamma)} \sum_{i=1}^{n_y} (\bar{z}_i + \boldsymbol{z}_i^{+T} \boldsymbol{\zeta}^+ + \boldsymbol{z}_i^{-T} \boldsymbol{\zeta}^-) \\ & \text{s.t.} \\ & \bar{z}_i + \boldsymbol{z}_i^{+T} \boldsymbol{\zeta}^+ + \boldsymbol{z}_i^{-T} \boldsymbol{\zeta}^- \leq (\bar{\alpha}_{ik}(\boldsymbol{x}) - \bar{\alpha}'_i)(\sum_{\ell<i} \hat{\alpha}_\ell \boldsymbol{e}_\ell)^T (\boldsymbol{\zeta}^+ - \boldsymbol{\zeta}^-) + \beta_{ik}(\boldsymbol{x}) - \bar{\gamma}/n_y , \quad \begin{array}{l} \forall (\boldsymbol{\zeta}^+, \boldsymbol{\zeta}^-) \in \mathcal{U}_\pm(\Gamma) \\ \forall i, \forall k . \end{array} \end{aligned}$$

Hence, once again Corollary 1 of Ardestani-Jaafari and Delage (2016) applies and allows us to complete the proof using exactly the same steps as for conditions i. and iii. □