Untying the Knot between a Stochastic Program and its Distribution

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Evidence that Managing an Investment Portfolio is Difficult

Value on Jan 1st 2009 of each dollar contribution made to the Caisse de Dépôts et de Placements

<table>
<thead>
<tr>
<th>Date of contribution</th>
<th>CDPQ</th>
<th>1-year guaranteed certificates</th>
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<tbody>
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<td>Jan 1st, 2008</td>
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<td>Jan 1st, 2007</td>
<td>$0.79</td>
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<td>Jan 1st, 2006</td>
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<td>Jan 1st, 2004</td>
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<td>Jan 1st, 2002</td>
<td>$1.22</td>
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<td>Jan 1st, 2001</td>
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Why are Financial Investments so Fragile?

Some reasons:

- A wide range of financial securities can be used for investment
- Securities have become very complex
- The risks involved are difficult to evaluate
- Limited knowledge of how the market will behave in the future
Fleet composition is a difficult decision problem:
- Fleet contracts are signed 10 to 20 years ahead of schedule.
- Many factors are still unknown at that time:
  e.g., passenger demand, fuel prices, etc.

Yet, most airline companies sign these contracts based on a single scenario of what the future may be.

Are airlines companies at risk of going bankrupt?
Let’s consider the stochastic programming problem:

$$\max_{x \in X} \mathbb{E}[u(h(x, \xi))]$$

where $x$ = decisions and $\xi$ = uncertain parameters.

Here, we assume that we know:

- The distribution of the random vector $\xi$
- A utility function that matches investor’s attitude to risk
Developing an accurate probabilistic model requires heavy engineering efforts:

- Need to collect enough observations
- Need to consult with experts of the field of practice
- Need to make simplifying assumptions

Yet, there are inherent pitfalls in the process:

- Expecting that a scenario might occur does not determine its probability of occurring
- Unexpected event (e.g., economic crisis) might occur
- The future might actually not behave like the past
Consider an urn with 30 blue balls and 60 other balls that are either red or yellow (you don’t know how many are red or yellow).

**Experiment 1:** Choose among the following two gambles
- Gamble A: If you draw a blue ball, then you win 100$
- Gamble B: If you draw a red ball, then you win 100$

**Experiment 2:** Choose among the following two gambles
- Gamble C: If you draw blue or yellow ball, then you win 100$
- Gamble D: If you draw red or yellow ball, then you win 100$

If you clearly prefer Gamble A & D, then you cannot be thinking in terms of expected utility.
Untying the SP from a Specific Distribution

- Let’s consider that the choice of $F$ is ambiguous.
- Use available information to define $\mathcal{D}$, such that $F \in \mathcal{D}$.
- We are faced with a multi-objective optimization problem:

$$\max_{x \in \mathcal{X}} \{ \mathbb{E}_F[u(h(x, \xi))] \} \quad \forall F \in \mathcal{D}$$

- Distributionally Robust Optimization values the lowest performing one.

$$\text{(DRSP)} \quad \max_{x \in \mathcal{X}} \min_{F \in \mathcal{D}} \mathbb{E}_F[u(h(x, \xi))]$$

- Recently, we found ways of solving some DRSP’s efficiently.
  [Popescu (2007), Bertsimas et al., Natarajan et al., Delage et al. (2010)].
- Possible to promote performance differently depending on $F$.
  [Föllmer et al. (2002), Li et al. (2011)].
Outline

1 Introduction

2 Distributionally Robust Optimization

3 Distributions Can Be Misleading

4 Value of Stochastic Modeling

5 Conclusion
Outline

1. Introduction
2. Distributionally Robust Optimization
3. Distributions Can Be Misleading
4. Value of Stochastic Modeling
5. Conclusion
Let’s make two assumptions about $\mathbb{E}[u(h(x, \xi))]$.

1. The utility function is piecewise linear concave:

$$u(y) = \min_{1 \leq k \leq K} a_k y + b_k,$$

2. The profit function is the maximum of a linear program with uncertainty limited to objective

$$h(x, \xi) := \max_y c_1^T x + \xi^T C_2 y$$

s.t. $A x + B y \leq b$
Resolving Distributional Set from Data

Question:

- We have in hand i.i.d. samples \( \{\xi_i\}_{i=1}^M \).
- We know that \( \mathbb{P}(\xi \in S) = 1 \) and \( S \subseteq B(0, R) \).
- We can estimate the mean and covariance matrix:

\[
\hat{\mu} = \frac{1}{M} \sum_{i=1}^{M} \xi_i \quad \hat{\Sigma} = \frac{1}{M} \sum_{i=1}^{M} (\xi_i - \hat{\mu})(\xi_i - \hat{\mu})^T
\]

What do we know about the distribution behind these samples?

Answer:

\[
\mathcal{D}(\gamma) = \left\{ F \left| \begin{array}{l}
\mathbb{P}(\xi \in S) = 1 \\
\| \mathbb{E} [\xi] - \hat{\mu} \|^2_{\Sigma^{-1/2}} \leq \gamma_1 \\
\mathbb{E} [(\xi - \hat{\mu})(\xi - \hat{\mu})^T] \succeq (1 + \gamma_2)\hat{\Sigma}
\end{array} \right. \right\}
\]

With prob. \( > 1 - \delta \) the distribution is contained in \( \mathcal{D}(\gamma) \) for some \( \gamma_1 = O\left(\frac{R^2}{M} \log(1/\delta)\right) \) and \( \gamma_2 = O\left(\frac{R^2}{\sqrt{M} \sqrt{\log(1/\delta)}}\right) \).
The DRSP is a SDP

- The DRSP problem with $\mathcal{D}(\gamma)$ is equivalent to

$$\max_{x, Q, q, r} \quad r - \left(\gamma_2 \hat{\Sigma} + \hat{\mu} \hat{\mu}^T\right) \cdot Q - \hat{\mu}^T q - \sqrt{\gamma_1} \| \hat{\Sigma}^{1/2} (q + 2Q\hat{\mu}) \|
\quad \text{s.t.} \quad r \leq \min_{\xi \in S} u(h(x, \xi)) + \xi^T q + \xi^T Q \xi \quad (\star)
\quad Q \succeq 0$$

- If $S = \text{polygon or ellipsoid}$, then DRSP equivalent to semi-definite program.
  E.g., when $S = \mathbb{R}^m$, Constraint (\star) can be replaced by

$$\begin{bmatrix} Q & (q + a_k C_2 y_k)/2 \\ (q + a_k C_2 y_k)^T/2 & a_k c_1^T x + b_k - r \end{bmatrix} \succeq 0, \ \forall \ k$$
If we are risk neutral we might not even need distribution information

Theorem

The solution of

\[
\max_{x \in \mathcal{X}} \mathbb{E}[h(x, \mu)]
\]

is optimal with respect to

\[
\max_{x \in \mathcal{X}} \inf_{F \in \mathcal{D}(\mu, \Psi)} \mathbb{E}_F[h(x, \xi)],
\]

for any set of convex functions \( \Psi \) with

\[
\mathcal{D}(\mu, \Psi) = \left\{ F \left| \begin{array}{l}
\mathbb{E}[\xi] = \mu \\
\mathbb{E}[\psi(\xi)] \leq 0, \ \forall \psi \in \Psi
\end{array} \right. \right\}.
\]
Outline

1. Introduction
2. Distributionally Robust Optimization
3. Distributions Can Be Misleading
4. Value of Stochastic Modeling
5. Conclusion
Let’s consider the case of portfolio optimization:

$$\max_{x \in \mathcal{X}} \min_{F \in \mathcal{D}} \mathbb{E}_F[u(\xi^T x)]$$

where $x_i$ is how much is invested in stock $i$ with future return $\xi_i$.

Does the robust solution perform better than a stochastic programming solution?

$$\mathcal{D} = \mathcal{D}(\gamma) \quad \text{vs.} \quad \mathcal{D} = \{ \hat{F} \}$$
30 stocks tracked over years 1992-2007 using Yahoo! Finance
10% and 90% percentiles are indicated periodically.
79% of time, the DRSP outperformed the exp. utility model
67% improvement on average using DRSP with $\mathcal{D}(\gamma)$
Divide a planar region into $K$ subregions, each serviced by a different vehicle, so that the total workload be most evenly distributed among the fleet.
Multi-Vehicle Routing on a Planar Region

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- Divide a planar region into \( K \) subregions, each serviced by a different vehicle, so that the total workload be most evenly distributed among the fleet.
Given $\mathcal{D}$, we partition so that the largest workload over the worst distribution of demand points is as small as possible

$$\min_{\{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_K\}} \sup_{F \in \mathcal{D}} \left\{ \max_i \mathbb{E}[TSP(\{\xi_1, \xi_2, \ldots, \xi_N\} \cap \mathcal{R}_i)] \right\},$$

A side product is to characterize for any partition what is a worst-case distribution of demand locations.
We simulated three partition schemes on a set of randomly generated parcel delivery problems where the territory needed to be divided into two regions and the demand is drawn from a mixture of truncated Gaussian distribution.
Robust partitions of the USA-Mexico border obtained using our branch & bound algorithm.
Consider the situation:

1. We know of a set $D$ such that $F \in D$
2. We have a candidate solution $x_1$ in mind
3. Is it worth developing a stochastic model: $D \rightarrow F$?
   
   (a) If yes, then develop a model & solve it
   (b) Otherwise, implement $x_1$

The Value of Stochastic Modeling ($VSM$) gives an optimistic estimate of the value of obtaining perfect information about $F$.

$$VSM(x_1) := \sup_{F \in D} \left\{ \max_{x_2} \mathbb{E}_F[h(x_2, \xi)] - \mathbb{E}_F[h(x_1, \xi)] \right\}$$

**Theorem**

*Unfortunately, evaluating $VSM(x_1)$ exactly is NP-hard in general.*
Theorem

If $S \subseteq \{\xi \mid \|\xi\|_1 \leq \rho\}$, an upper bound can be evaluated in $O(d^{3.5} + d T_{DCP})$ using:

$$UB(x_1, \bar{y}_1) := \min_{s, q} \ s + \mu^T q$$

s.t. $s \geq \alpha(\rho e_i) - \rho e_i^T q$, $\forall \ i \in \{1, \ldots, d\}$

$s \geq \alpha(-\rho e_i) + \rho e_i^T q$, $\forall \ i \in \{1, \ldots, d\}$,

where $\alpha(\xi) = \max_{x_2} h(x_2, \xi) - h(x_1, \xi; \bar{y}_1)$.

- $UB$ only uses information about $\mu$ and $S$
- $UB$ simplifies the structure of $S$
- $UB$ assumes the candidate decision $y_1$ cannot adapt to $\xi$
The fleet composition problem is a stochastic mixed integer LP

\[
\text{max. } \mathbb{E} \left[ - \mathbf{o}^\top \mathbf{x} + h(\mathbf{x}, \mathbf{p}, \mathbf{c}, \mathbf{L}) \right],
\]

with \( h(\mathbf{x}, \mathbf{p}, \mathbf{c}, \mathbf{L}) := \)

\[
\max_{z \geq 0, y \geq 0, w} \sum_k \left( \sum_i \tilde{p}_i^k w_i^k - \tilde{c}_k (z_k - x_k)^+ + \tilde{L}_k (x_k - z_k)^+ \right)
\]

s.t. \( w_i^k \in \{0, 1\}, \forall k, \forall i \) \& \( \sum_k w_i^k = 1, \forall i \) \} \text{ Cover}

\[
y_g \in \text{in}(v) + \sum_{i \in \text{arr}(v)} w_i^k = y_g \in \text{out}(v) + \sum_{i \in \text{dep}(v)} w_i^k, \forall k, \forall v \} \text{ Balance}
\]

\[
z_k = \sum_{v \in \{v\mid \text{time}(v) = 0\}} (y_g \in \text{in}(v) + \sum_{i \in \text{arr}(v)} w_i^k), \forall k
\]
We experimented with three test cases:

1. 3 types of aircrafts, 84 flights, $\sigma\tilde{p}_i/\mu\tilde{p}_i \in [4\%, 53\%]$
2. 4 types of aircrafts, 240 flights, $\sigma\tilde{p}_i/\mu\tilde{p}_i \in [2\%, 20\%]$
3. 13 types of aircrafts, 535 flights, $\sigma\tilde{p}_i/\mu\tilde{p}_i \in [2\%, 58\%]$

Results:

<table>
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<tr>
<th>Test cases</th>
<th>Computation Times</th>
<th>Upper bound for VSM</th>
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<tbody>
<tr>
<td></td>
<td>DCP</td>
<td>SP (100 scen.)</td>
</tr>
<tr>
<td>#1</td>
<td>0.6 s</td>
<td>3 min</td>
</tr>
<tr>
<td>#2</td>
<td>1 s</td>
<td>14 min</td>
</tr>
<tr>
<td>#3</td>
<td>5 s</td>
<td>21 h</td>
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Conclusions:

- It’s wasteful to invest more than 7% of profits in extra info
Conclusion & Future Work

- Many forms of the DRSP are tractable
- Some actually reduce to the DCP
- Thinking we know the distribution can be misleading
- Knowing the actual distribution might not help that much
- There are tools that help estimate how much the true distribution is worth

Open questions:
- Can tractable DRSP be made consistent?
- Can DRSP be extended to multi-objective problems?
- How to deal with ambiguity about one’s utility function?
Bibliography


Questions & Comments ...

... Thank you!