Optimal Hedging Strategies With an Application to Hedge Fund Replication

Alexandre Hocquard, Nicolas Papageorgiou*, Bruno Rémillard, HEC Montréal

*Finance Department, HEC Montréal, 3000 Cote Sainte-Catherine, Montreal, QC, H3T 2A7, Canada, e-mail: nicolas.papageorgiou@hec.ca

Abstract

The derivation of the bi-variate Payoff Distribution model by Kat and Palaro (2005) represents an interesting contribution to the performance evaluation and asset pricing literature. Nonetheless, their approach for evaluating the function is significantly flawed. Recently, Papageorgiou *et al.* (2007) have proposed a much more robust approach to modeling the marginal distributions and copula functions, and also extend the results of Schweizer (1995) to evaluate the model and derive an optimal dynamic trading (hedging) strategy. In this paper, we will discuss the technical challenges of implementing a multi-variate extension of Dybvig (1988) model and discuss the possible solutions.

1 Introduction

Over the last couple of years, considerable attention paid within the hedge fund industry to the development replicating strategies. Many of the large banks have launched beta replication funds that attempt to use a portfolio of liquid assets to replicate the time-series properties of various hedge fund strategies.[1] The tracking portfolio generally consists in exposure to market, credit and liquidity premia. However, the replicating portfolio may consist of assets that are not necessarily employed by managers (e.g., high yield bonds may explain exposure of hedge equity to liquidity risk).

An interesting alternative replication method was proposed by Amin and Kat (2003) and more recently extended by Kat and Palaro (2005). Based on the Payoff Distribution Model put forth by Dybvig (1988), the authors attempt to replicate hedge fund returns not by identifying the return generating betas, but identifying a systematic trading strategy that can be used to generate the distribution of the hedge fund returns. Kat and Palaro (2005) show that for most hedge funds, their statistical properties can be replicated by investing in an alternative dynamic strategy.

The derivation of the bivariate Payoff Distribution Model by Kat and Palaro (2005) represents an interesting contribution to the performance evaluation and asset pricing literature. The implementation proposed by Kat and Palaro is however subject to several shortcomings and inconsistencies. In this paper we will address these problems and propose some techniques for overcoming these issues.

2 The Payoff Function

In Kat and Palaro (2005), the authors show that given two risky assets $S^{(1)}$ and $S^{(2)}$, it is possible to "reproduce" the statistical properties of the joint composed returns $R^{(1)}_{0,T} = \log(S^{(1)}_T/S^{(1)}_0)$ and $R^{(3)}_{0,T} = \log(S^{(3)}_T/S^{(3)}_0)$, in the sense that there exists a function g such that the joint distribution of $R^{(1)}_{0,T}$ and $g(R^{(1)}_{0,T}, R^{(2)}_{0,T})$ is the same as the joint distribution of $R^{(1)}_{0,T}$ and $R^{(3)}_{0,T}$. Note that one does not replicate the value of $R^{(3)}_{0,T}$ at period T, but instead one wants to imitate its distribution properties like its expectation, volatility, skewness, kurtosis, as well as dependence measures with respect to $R^{(1)}_{0,T}$ such as Pearson and Spearman correlations to name a few.

The payoff's return function g is easily shown to be calculable using the marginal distribution functions F_1, F_2 and F_3 of $S^{(1)}_T, S^{(2)}_T, S^{(3)}_T$, and the copulas $C_{1,2}$ and $C_{1,3}$ associated respectively with the joints returns $(R^{(1)}_{0,T}, R^{(2)}_{0,T})$ and $(R^{(1)}_{0,T}, R^{(3)}_{0,T})$. For details on its derivations see Kat and Palaro (2005). The exact expression for g is given by

$$g(x, y) = Q\left\{x, P\left(R^{(2)}_{0,T} \le y | R^{(1)}_{0,T} = x\right)\right\}, \qquad (1)$$

where $Q(x, \alpha)$ is the order α quantile of the conditional law of $R^{(3)}_{0,T}$ given $R^{(1)}_{0,T} = x$, i.e., for any $\alpha \in (0, 1)$, $q(x, \alpha)$ satisfies

$$P\left\{R^{(3)}_{0,T} \le Q(x, \alpha) | R^{(1)}_{0,T} = x\right\} = \alpha.$$

Using properties of copulas, e.g. Nelsen (1999), the conditional distributions can be expressed in terms of the margins and the associated copulas.

$$P\left(R_{0,T}^{(2)} \leq y | R_{0,T}^{(1)} = x\right) = \frac{\partial}{\partial u} \mathcal{C}_{1,2}(u, v)\Big|_{u=F_1(x), v=F_2(y)}.$$

Once the function has been calculated all that remains is to find the trading strategy that will allow to replicate the function. In essence, we can view the function as an option that cannot be traded, so we need to replicate the payoff of the option with the greatest possible precision by trading the underlying securities.

3 Replication and the Shortcomings of the Kat-Palaro Approach

There are three steps in the replication procedure.

- Modelling part:
  - Estimation of the parameters of the marginal distribution functions F_1, F_2 and F_3 of $S_T^{(1)}, S_T^{(2)}, S_T^{(3)}$, and the copulas $\mathcal{C}_{1,2}$ and $\mathcal{C}_{1,3}$ associated respectively with the joints returns $(R_{0,T}^{(1)}, R_{0,T}^{(2)})$ and $(R_{0,T}^{(1)}, R_{0,T}^{(3)})$.
- Calculate the payoff function g.
- Replication part:
  - Choose an appropriate replication method;
  - Find the initial amount v_0 to be invested in the portfolio and find an hedging strategy φ.

3.1 Modeling issues

The correct calculation of the payoff function relies therefore on the precise modeling of the statistical properties of our three assets. The marginal distributions F_1, F_2 and F_3 must be capable of capturing the necessary skewness and kurtosis, and a proper empirical test must be implemented in order to select the two copulas $\mathcal{C}_{1,2}$ and $\mathcal{C}_{1,3}$. Any mis-specification of the statistical properties will induce an error in the calculation of the payoff function g, which, in turn, will not capture the statistical properties of $R_{0,T}^{(3)}$. Kat and Palaro (2005) use the Gaussian, Student and Johnson distributions to model the monthly returns of the three assets and five copula functions (Gaussian, Student, Frank, Gumbel and symmetrized Joe-Clayton) to model the dependence. The estimation and choice of marginal distribution and copula is performed using the Inference for Margins (IFM) method.

There are two significant shortcomings related to the modeling approach proposed by Kat and Palaro (2005). The first issue relates to the aggregation properties of the distributions and copula functions, and represents a fundamental flaw in the modeling approach. The second issue is also not trivial and relates to the choice of estimation technique.

The main flaw in the Kat and Palaro (2005) model has to do with the distribution of the returns $R_1, \ldots, R_T$ versus the distribution of $R_{0,T}$. In their paper, Kat and Palaro (2005) start by fixing the law of the monthly returns, distribution functions F_1, F_2, F_3 and the copulas $\mathcal{C}_{1,2}, \mathcal{C}_{1,3}$, and then solve for the corresponding daily hedging strategy for assets $S^{(1)}$ and $S^{(2)}$. The compatibility problem between the law of the daily returns and monthly returns is not addressed by the authors. According to Sklar's theorem (Sklar, 1959), the law of the bivariate vector $R_{0,T}$ is determined by F_1, F_2 and $\mathcal{C}_{1,2}$. However, the joint law of the returns $(R_t)_{t=1}^T$ must be compatible with the relation

$$R_{0,T} = \sum_{t=1}^{T} R_t. \tag{2}$$

Let's consider, for the sake of simplicity, that returns are independent and identically distributed. In the bivariate Gaussian case, it is easy to find the law of the returns $(R_t)_{t=1}^T$ given the law of $R_{0,T}$. In fact, even if the marginal distribution of $R_{0,T}^{(1)}$ and $R_{0,T}^{(2)}$ are Gaussian and their copula $\mathcal{C}_{1,2}$ is not Gaussian, the margins of $R_t^{(1)}$ and $R_t^{(2)}$ are Gaussian. However, there is no known way to find out what the common copula of the R_t's should be so that the copula of the sum match the copula $\mathcal{C}_{1,2}$. Although copula provide us with much flexibility in terms of modeling the dependence, there is however no proof to this day that the statistical properties of copula functions are divisible. This compatibility condition is not a trivial matter. In fact, if for any T, the relation (2) is satisfied with independent and identically distributed returns $(R_t)_{t=1}^T$, then the law of $R_{0,T}$ must be infinitely divisible. Such laws can be characterized completely (see Barndorff-Nielsen *et al.* (2001) or Sato (1999)). For example, it is known that the univariate Student distribution is infinitely divisible, but the common law of the associated returns $(R_t)_{t=1}^T$ satisfying (2) is not known. Note that Johnson's law, proposed in Kat and Palaro (2005), is not infinitely divisible. Therefore, it should not serve as a model for the distribution of $R_{0,T}^{(1)}$ or $R_{0,T}^{(2)}$ if the daily returns are assumed to be independent.

The lesser of the two problems pertains to the choice of estimation technique. IFM is a two-stage estimation process: first the marginal distributions are estimated and then these distributions are used in order to calculate the parameters of the copula. Kim *et al.* (2007) show that an inappropriate choice of models for the margins may have detrimental effects on the estimation of the dependence parameter per se. A much more robust method consists of separating the estimation for the margins and the dependence. Ideally the estimation of the dependence should rely on normalized ranks and be independent of the marginal distributions. For a detailed description see Genest *et al.* (1995).

3.1.1 Overcoming the aggregation problems

In order to deal with the compatibility restriction, instead of estimating the law of the monthly returns $R_{0,T}$ for assets $S^{(1)}$ and $S^{(2)}$, it is preferable to take the opposite point of view, by first determining a model for the daily returns $(R_t)_{t=1}^T$, and then solving for the associated law for the composed bivariate return $R_{0,T}$. The important issue is select bivariate laws whose aggregation properties are known. A good candidate for the law of the returns R_t is a mixture of bivariate Gaussian distributions. It is easy to check that the law of $R_{0,T}$ will then be also a Gaussian mixture. Properties of Gaussian mixtures, as well as estimation and goodness-of-fit are treated in Papageorgiou *et al.* (2007). We do not need to concern ourselves with the distribution of asset $S^{(3)}$ since it is not used in the trading strategy.

A concern in the modeling of the daily returns can be presence of serial correlation in the daily time series. One interesting extension of Papageorgiou *et al.* (2007) would be to be to the model joint returns of assets $S^{(1)}$ and $S^{(2)}$ as a mixture of bivariate Gaussian distribution with a Markovian dependence in the mixtures. One could also consider mixture of bivariate GARCH processes. The aggregation properties and estimation of multi-variate mixtures of GARCH processes have been studied by Hafner and Rombouts (2007).

3.2 Hedging issues

Having modeled the return distributions and dependence structures, we can then calculate the payoff function g. The final step is to find a dynamic trading strategy that allows us to best approximate this function. The hedging strategy proposed by Kat and Palaro (2005) is quite simple. They use a trinomial approach proposed by He (1990) even though the law of the (daily) returns

$$R_t = \left\{\log\left(S_t^{(1)}/S_{t-1}^{(1)}\right), \log\left(S_t^{(2)}/S_{t-1}^{(2)}\right)\right\}^{\mathrm{T}}$$

is not necessarily Gaussian.

In their calculations they implemented the technique of Boyle and Lin (1997), a trinomial approach that incorporates transactions costs. This approach is clearly inefficient, specially since the distributions of the traded assets $S^{(1)}$ and $S^{(2)}$, and the hedge fund $S^{(3)}$ are clearly not Gaussian. In order to get rid of this inconsistency which is common in option pricing, Papageorgiou *et al.* (2007) propose an alternative methodology adapted from American option pricing techniques. The authors extend the results of Schweizer (1995) by selecting the portfolio (v_0, φ) such as to minimize the (square) root mean square hedging error (RMSHE)

$$\sqrt{E[\beta_T^2\{V_T(v_0,\varphi) - C_T\}^2]},$$

where β_T is the discount factor and φ is a dynamic replication strategy. The value, at period t, of the portfolio defined by the initial value v_0 and strategy φ is denoted by $V_t(v_0, \varphi)$. Note that there is no "risk-neutral" evaluation involved, all calculations are carried out under the objective probability measure.

3.2.1 Optimal hedging

Suppose that $(\Omega, P, \mathcal{F})$ is a probability space with filtration $\mathbb{F} = \{\mathcal{F}_0, \ldots, \mathcal{F}_T\}$, under which the stochastic processes are defined. Assume that the price process S_t is d-dimensional, i.e. $S_t = (S_t^{(1)}, \ldots, S_t^{(d)})$.

A dynamic replicating strategy can be described by a (deterministic) initial value v_0 and a sequence of random weight vectors $\varphi = (\varphi_t)_{t=0}^T$, where for any $j = 1, \ldots, d$, $\varphi_t^{(j)}$ denotes the number of parts of assets $S^{(j)}$ invested during period $(t-1, t]$. Because φ_t may depend only on the values $S_0, \ldots, S_{t-1}$, the stochastic process φ_t is assumed to be predictable. Initially, $\varphi_0 = \varphi_1$, and the portfolio initial value is v_0. It follows that the amount initially invested in the non risky asset is $v_0 - \sum_{j=1}^d \varphi_1^{(j)} S_0^{(j)} = v_0 - \varphi_1^{\mathrm{T}} S_0$.

Since the hedging strategy must be self-financing, it follows that for all $t = 1, \ldots, T$,

$$\beta_t V_t(v_0,\varphi) - \beta_{t-1}V_{t-1}(v_0,\varphi) = \varphi_t^{\mathrm{T}}(\beta_t S_t - \beta_{t-1}S_{t-1}). \tag{3}$$

Using the self-financing condition (3), it follows that

$$\beta_T V_T = \beta_T V_T(v_0,\varphi) = v_0 + \sum_{t=1}^T \varphi_t^{\mathrm{T}}(\beta_t S_t - \beta_{t-1}S_{t-1}). \tag{4}$$

The replication strategy problem for a given payoff C is thus equivalent to finding the strategy (v_0, φ) so that the hedging error

$$G_T(v_0,\varphi) = \beta_T V_T(v_0,\varphi) - \beta_T C \tag{5}$$

is as small as possible. Here, the RMSHE measures the quality of replication. It is therefore natural to suppose that the prices $S_t^{(j)}$ have finite second moments. We further assume that the hedging strategy φ satisfies a similar property, namely that for any $t = 1, \ldots, T$, $\varphi_t^{\top}(\beta_t S_t - \beta_{t-1}S_{t-1})$ have finite second moments. Note that these two technical conditions were also made by Schweizer (1995).

For simplicity, set $\Delta_t = S_t - E(S_t|\mathcal{F}_{t-1})$, $t = 1, \ldots, T$. Under the above moment conditions, the conditional covariance matrix Σ_t of Δ_t exists and is given by

$$\Sigma_t = E\left\{\Delta_t\Delta_t^{\top}|\mathcal{F}_{t-1}\right\}, \quad 1 \le t \le T. \tag{6}$$

In Schweizer (1995), the author treats the case $d = 1$ and assumes a restrictive boundedness condition. Here, in contrast, we treat the general d-dimensional case and we only suppose that Σ_t is invertible for all $t = 1, \ldots, T$. This was implicitly part of the boundedness condition of Schweizer (1995).

If Σ_t is not invertible for some t, there would exists a $\varphi_t \in \mathcal{F}_{t-1}$ such that $\varphi_t^{\top} S_t = \varphi_t^{\top} E(S_t|\mathcal{F}_{t-1})$, that is, $\varphi_t^{\top} S_t$ is predictable. Our assumption can be interpreted as saying that the genuine dimension of the assets is d.

3.2.2 Difference between optimal hedging and hedging under Black-Scholes setting

To compare the two methods, simply take $T = 1$, $\beta_T = 1$, and $d = 1$. In this case, the solution for optimal hedging yields $\varphi^* = \mathrm{Cov}\{\Delta S_1, C(S_1)\}/\mathrm{Var}(\Delta S_1)$, where $\Delta S_1 = S_1 - S_0$, and $v_0^* = E\{C(S_1)\} - \varphi^* E(\Delta S_1)$.

For the Black-Scholes setting, $v_0^{BS} = E\left\{C\left(S_0 e^{\sigma Z - \sigma^2/2}\right)\right\}$ and $\varphi^{BS} = E\left\{e^{\sigma Z-\sigma^2/2}C'\left(S_0 e^{\sigma Z-\sigma^2/2}\right)\right\}$, with $\sigma^2 = \mathrm{Var}\{\log(S_1/S_0)\}$, where $Z \sim N(0,1)$, provided C is differentiable. See, e.g., Broadie and Glasserman (1996).

In general, $\varphi^* \neq \varphi^{BS}$ and $v_0^* \neq v_0^{BS}$, so

$$E[\{V_1(v_0^*,\varphi^*) - C(S_1)\}^2] < E\left[\{V_1(v_0^{BS},\varphi^{BS}) - C(S_1)\}^2\right].$$

For an analysis of the (discrete) hedging error in a Black-Scholes setting, see, e.g., Wilmott (2006).

3.2.3 Hedging Error Comparison

To illustrate the advantage of the optimal hedging strategy proposed in Papageorgiou *et al.* (2007), we compare the mean hedging error and the

RMSHE as defined in equation (5) for the optimal hedging and for the Kat-Palaro approach. For this example, we specify assets $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ as follows:

- Asset $S^{(1)}$ is a proxy for the typical institutional Canadian pension fund as described in *Benefits Canada Review (May 2007)*.
- Asset $S^{(2)}$ is a diversified portfolio of typical market exposures, specifically global equity indices, credit indices and commodity indices
- Asset $S^{(3)}$ that is being replicated is chosen to be gaussian distribution with an annual volatility of 12%.

We model bivariate daily and monthly distributions of assets $S^{(1)}$ and $S^{(2)}$ over the period from 2000 to 2007 using normal mixtures, as detailed in Papageorgiou *et al.* (2007). This leads to 7 regimes for the daily mixture and 2 regimes for the monthly mixture. We do not specify the required dependence between $S^{(3)}$ and $S^{(1)}$, instead we run the hedging comparison for different levels of dependence between the two assets. More precisely, we allow Kendall's Tau to vary from -0.9 to 0.9 for three different copulas (Gaussian, Clayton and Frank) and measure the impact of this dependency variable between $S^{(1)}$ and $S^{(3)}$ on hedging error measures. To compare the optimal hedging replication method and the Kat-Palaro method, 10 000 scenarios of 22 daily returns (1 trading month) were simulated for the assets $S^{(1)}$ and $S^{(2)}$. For each scenario, the terminal value V_T of the portfolio was computed and the hedging error is calculated. The plots of the hedging errors are presented below.

The results lend strong support to the hedging approach put forth in Papageorgiou *et al.* (2007). Hedging Errors for the "Optimal Hedging" algorithm are centered on 0 with a low sensitivity to Kendall's Tau as well as to the type of copula. The Kat-Palaro algorithm is considerably more sensitive to the level of dependence (Kendall's tau) and copula family. This is a direct result of their approach being nested in the Black-Scholes setting and can lead to large hedging errors. It is also important to note that the Optimal Hedging approach systematical produces smaller Root Mean Square Hedging Errors (RMSHE) providing further validation of the Papageorgiou *et al.* (2007) approach.

4 Conclusion

In the paper, we have discussed some of the challenges that one is confronted with in implementing the bivariate Payoff Distribution Model proposed by Kat and Palaro (2005). We exposed some of the flaws in the modeling and the dynamic trading strategy, and proposed some techniques for overcoming these inconsistencies. Finally, we showed that the hedging algorithm proposed in Papageorgiou *et al.* (2007) provides a more precise replication of the payoff function that the Black-Scholes approach put forth by Kat and Palaro (2005).

What remains to be seen is how well these statistical replication techniques fare in practice. Desjardins Global Asset Management should soon be able to provide some insight into this issue. They have been

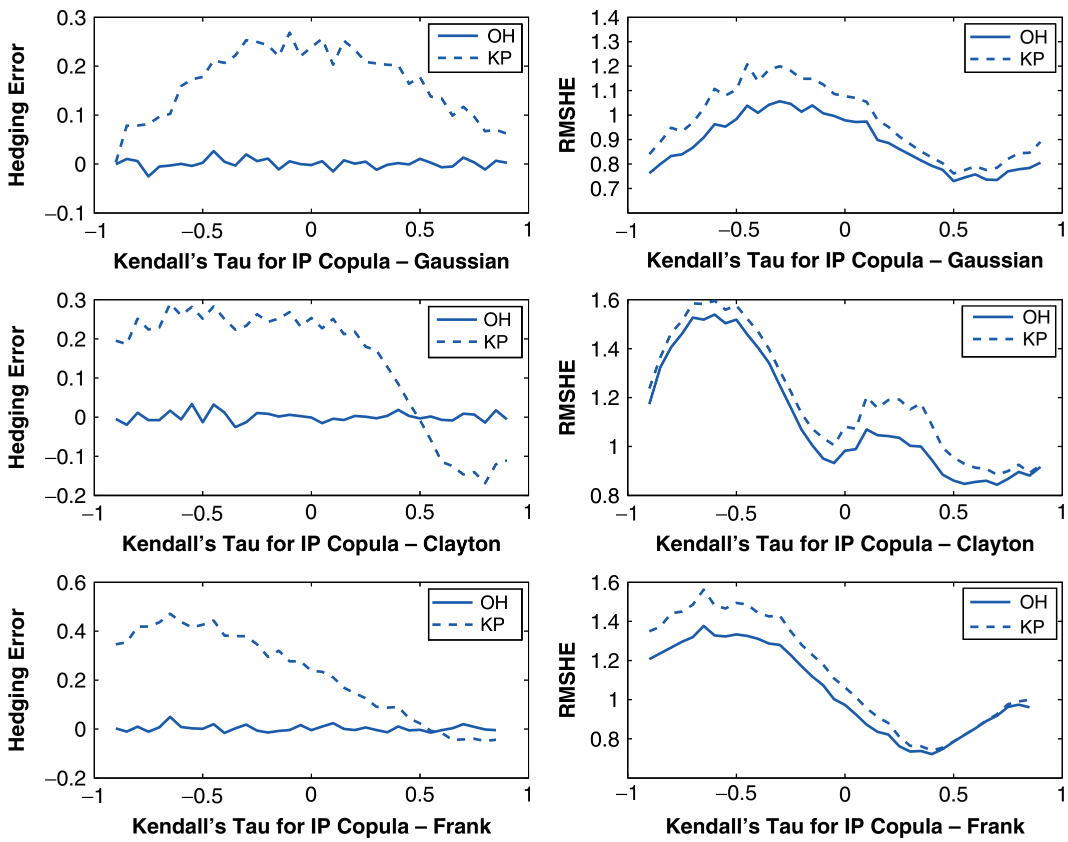

Figure 1: Hedging error measures.

working with the authors of Papageorgiou *et al.* (2007) and have recently launched the first statistical replication fund that is open to investors.

Acknowledgements

This work is supported in part by the Fonds pour la formation de chercheurs et l'aide à la recherche du Gouvernement du Québec, by the Fonds québécois de la recherche sur la société et la culture, by the Natural Sciences and Engineering Research Council of Canada and by Desjardins Global Asset Management.

FOOTNOTES & REFERENCES

1 ML Factor Index, GS Absolute Return Tracker, Partners Group AB Program, JPM AB Index

Amin, G. and Kat, H. (2003). Hedge fund performance 1990–2000: Do the "money machines" rellay add value. *Journal of Financial and Quantitative Analysis,* 38(2):251–275.

Barndorff-Nielsen, O., Mikosch, T., and Resnick, S., editors (2001). Levy Processes: Theory and Applications. Birkhäiser, Boston.

Boyle, P. and Lin, X. (1997). Valuation of options on several risky assets when there are trabnsactions costs. In Boyle, P., Pennacchi, G., and Richken, P., editors, *Advances in Futures and Options Research*, volume 9, pages 111–129. JAI Press.

Broadie, M. and Glasserman, P. (1996). Estimating security price derivatives using simulation. *Management Science*, 42:260–285.

Dybvig, P. (1988). Distributional analysis of portfolio choice. *The Journal of Business,* 61(3):369–393.

Genest, C., Ghoudi, K., and Rivest, L.-P. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika,* 82:543–552.

Hafner, C. and Rombouts, J. (2007). Estimation of temporally aggregated multivariate GARCH models. *Journal of Statistical Computation and Simulation,* 77:629–650.

He, H. (1990). Convergence from discrete to continuous-time contingent claims price. *Review of Financial Studies,* 3:523–546.

Kat, H. and Palaro, H. (2005). Who needs hedge funds? A copula-based approach to hedge fund return replication. Technical report, Cass Business School, City University.

Kim, G., Silvapulle, M. J., and Silvapulle, P. (2007). Comparison of semiparametric and parametric methods for estimating copulas. *Comm. Statist. Simulation Comput.,* 51:2836–2850.

Nelsen, R. B. (1999). *An introduction to copulas*, volume 139 of *Lecture Notes in Statistics.* Springer-Verlag, New York.

Papageorgiou, N., Rémillard, B., and Hocquard, A. (2007). Replicating the properties of hedge fund returns. Technical Report 1, DGAM-HEC Alternative Invesments Research.

Sato, K.-I. (1999). *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge. Translated from the 1990 Japanese original, Revised by the author.

Schweizer, M. (1995). Variance-optimal hedging in discrete time. *Math. Oper. Res.,* 20(1):1–32.

Sklar, M. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris,* 8:229–231.

Wilmott, P. (2006). *Paul Wilmott on Quantitative Finance*, volume 3. John Wiley & Sons.