Testing for Restricted Stochastic Dominance

by

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Abstract

Asymptotic and bootstrap tests are studied for testing whether there is a relation of stochastic dominance between two distributions. These tests have a null hypothesis of nondominance, with the advantage that, if this null is rejected, then all that is left is dominance. This also leads us to define and focus on *restricted* stochastic dominance, the only empirically useful form of dominance relation that we can seek to infer in many settings. One testing procedure that we consider is based on an empirical likelihood ratio. The computations necessary for obtaining a test statistic also provide estimates of the distributions under study that satisfy the null hypothesis, on the frontier between dominance and nondominance. These estimates can be used to perform bootstrap tests that turn out to provide much improved reliability of inference compared with the asymptotic tests so far proposed in the literature.

Keywords: Stochastic dominance, empirical likelihood, bootstrap test

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1. Introduction

Consider two probability distributions, A and B, characterised by cumulative distribution functions (CDFs) F_A and F_B. In practical applications, these distributions might be distributions of income, before or after tax, wealth, or of returns on financial assets. Distribution B is said to dominate distribution A stochastically at first order if, for all z in the union of the supports of the two distributions, $F_A(z) \geq F_B(z)$. If B dominates A, then it is well known that expected utility and social welfare are greater in B than in A for all utility and social welfare functions that are symmetric and monotonically increasing in returns or in incomes (and lower for all poverty indices that are symmetric and monotonically decreasing in incomes). These are powerful orderings of the two distributions, and it is therefore not surprising that a considerable empirical literature has sought to test for such dominance in the last decades.

Until now, the most common way to test whether there is stochastic dominance, on the basis of samples drawn from the two populations A and B, is to posit a null hypothesis of dominance, and then to study test statistics that may or may not lead to rejection of this hypothesis. Non-rejection of dominance unfortunately does not enable one to accept dominance, the usual outcome of interest. It may thus be preferable, from a logical point of view, to posit a null of *non*-dominance, since, if we succeed in rejecting this null, the only other possibility is dominance, and we may therefore draw the conclusion of dominance.

We adopt this latter standpoint in this paper. We find that it leads to testing procedures that are actually simpler to implement than conventional procedures in which the null is dominance. The simplest procedure for testing nondominance was proposed originally by Kaur, Prakasa Rao, and Singh (1994) (henceforth KPS) for continuous distributions A and B, and a similar test was proposed in an unpublished paper by Howes (1993) for discrete distributions. In this paper, we develop an alternative procedure, based on an empirical likelihood ratio statistic. It turns out that this statistic is always numerically very similar to the KPS statistic in all the cases we consider. However, the empirical likelihood approach produces as a by-product a set of probabilities that can be interpreted as estimates of the population probabilities under the assumption of nondominance.

These probabilities make it possible to set up a bootstrap data-generating process (DGP) which lies on the frontier of the null hypothesis of nondominance. We show that, on this frontier, both the KPS and the empirical likelihood statistics are asymptotically pivotal, by which we mean that they have the same asymptotic distribution for all configurations of the population distributions that lie on the frontier. A major finding of this paper is that bootstrap tests that make use of the bootstrap DGP we define yield much more satisfactory inference than tests based on the asymptotic distributions of the statistics.

The paper also shows that it is not possible with continuous distributions to reject nondominance in favour of dominance over the entire supports of the distributions. Accepting dominance is empirically sensible only over *restricted* ranges of incomes and returns. This necessitates a recasting of the usual theoretical links between stochastic dominance relationships and orderings in terms of poverty, social welfare and expected utility. It also highlights better why a non-rejection of the usual null hypothesis of unrestricted dominance *cannot* be interpreted as an acceptance of dominance, since unrestricted dominance can never be inferred from continuous data.

In Section 2, we investigate the use of empirical likelihood methods for estimation of distributions under the constraint that they lie on the frontier of nondominance, and develop the empirical likelihood ratio statistic. The statistic is a minimum over all the points of the realised samples of an empirical likelihood ratio that can be defined for all points z in the support of the two distributions. In Section 3 we recall the KPS statistic, which is defined as a minimum over z of a t statistic, and show that the two statistics are locally equivalent for all z at which $F_A(z) = F_B(z)$. Section 4 shows why it turns out to be impossible to reject the null of nondominance when the population distributions are continuous in their tails. Some connections between this statistical fact and ethical considerations are explored in Section 5, and the concept of restricted stochastic dominance is introduced. In Section 6, we discuss how to test restricted stochastic dominance, and then, in Section 7 we develop procedures for testing the null of nondominance, in which we are obliged to censor the distributions, not necessarily everywhere, but at least in the tails. In that section, we also show that, for configurations of nondominance that are *not* on the frontier, the rejection probabilities of tests based on either of our two statistics are no greater than they are for configurations on the frontier. This allows us to restrict attention to the frontier, knowing that, if we can control Type I error there by choice of an appropriate significance level, then the probabilty of Type I error in the interior of the null hypothesis is no greater than that on the frontier. We are then able to show that the statistics are asymptotically pivotal on the frontier. Section 8 presents the results of a set of Monte Carlo experiments in which we investigate the rejection probabilities of both asymptotic and bootstrap tests, under the null and under some alternative setups in which there actually is dominance. We find that bootstrapping leads to very considerable gains in the reliability of inference. Conclusions and some related discussion are presented in Section 9.

2. Stochastic Dominance and Empirical Likelihood

Consider two distributions A and B, characterised by the cumulative distribution functions (CDFs) F_A and F_B. Distribution B stochastically dominates A at first order if, for all x in the union U of the supports of the two distributions, $F_A(x) \geq F_B(x)$. In much theoretical writing, this definition also includes the condition that there should exist at least one x for which $F_A(x) > F_B(x)$ strictly. Since in this paper we are concerned with statistical issues, we ignore this distinction between weak and strong dominance since no statistical test can possibly distinguish between them.

Suppose now that we have two samples, one each drawn from the distributions A and B. We assume for simplicity that the two samples are independent. Let N_A

and N_B denote the sizes of the samples drawn from distributions A and B respectively. Let Y^A and Y^B denote respectively the sets of (distinct) realisations in samples A and B, and let Y be the union of Y^A and Y^B. If, for $K = A, B$, y_t^K is a point in Y^K, let the positive integer n_t^K be the number of realisations in sample K equal to y_t^K. This setup is general enough for us to be able to handle continuous distributions, for which all the $n_t^K = 1$ with probability 1, and discrete distributions, for which this is not the case. In particular, discrete distributions may arise from a discretisation of continuous distributions. The empirical distribution functions (EDFs) of the samples can then be defined as follows. For any $z \in U$, we have

$$\hat{F}_K(z) = \frac{1}{N_K} \sum_{y_t^K \le z} n_t^K.$$

If it is the case that $\hat{F}_A(y) \ge \hat{F}_B(y)$ for all $y \in Y$, we say that we have first-order stochastic dominance of A by B in the sample.

In order to conclude that B dominates A with a given degree of confidence, we require that we can reject the null hypothesis of nondominance of A by B with that degree of confidence. Such a rejection may be given by a variety of tests. In this section we develop an empirical likelihood ratio statistic on which a test of the null of non-dominance can be based. As should become clear, it is relatively straightforward to generalise the approach to second and higher orders of dominance, although solutions such as those obtained analytically here would then need to be obtained numerically.

For a given sample, the "parameters" of the empirical likelihood are the probabilities associated with each point in the sample. The empirical loglikelihood function (ELF) is then the sum of the logarithms of these probabilities. If as above we denote by n_t the multiplicity of a realisation y_t, the ELF is $\sum_{y_t \in Y} n_t \log p_t$, where Y is the set of all realisations, and the p_t are the "parameters". If there are no constraints, the ELF is maximised by solving the problem

$$\max_{p_t} \sum_{y_t \in Y} n_t \log p_t \quad \text{subject to} \quad \sum_{y_t \in Y} p_t = 1.$$

It is easy to see that the solution to this problem is $p_t = n_t/N$ for all t, N being the sample size, and that the maximised ELF is $-N \log N + \sum_t n_t \log n_t$, an expression which has a well-known entropy interpretation.

With two samples, A and B, using the notation given above, we see that the probabilities that solve the unconstrained maximisation problem are $p_t^K = n_t^K/N_K$ for $K = A, B$, and that the maximised ELF is

$$-N_A \log N_A - N_B \log N_B + \sum_{y_t^A \in Y^A} n_t^A \log n_t^A + \sum_{y_t^B \in Y^B} n_t^B \log n_t^B. \tag{1}$$

Notice that, in the continuous case, and in general whenever $n_t^K = 1$, the term $n_t^K \log n_t^K$ vanishes.

The null hypothesis we wish to consider is that B does not dominate A, that is, that there exists at least one z in the interior of U such that $F_A(z) \leq F_B(z)$. We need z to be in the interior of U because, at the lower and upper limits of U, which is the union of the supports of the two distributions, we always have $F_A(z) = F_B(z)$, since both are either 0 or 1. In the samples, we exclude the smallest and greatest points in the set Y of realisations, for the same reason. We write Y° for the set Y without its two extreme points. If there is a $y \in Y^\circ$ such that $\hat{F}_A(y) \leq \hat{F}_B(y)$, there is nondominance in the samples, and, in such cases, we plainly do not wish to reject the null of nondominance. This is clear in likelihood terms, since the unconstrained probability estimates satisfy the constraints of the null hypothesis, and so are also the constrained estimates.

If there is dominance in the samples, then the constrained estimates must be different from the unconstrained ones, and the empirical loglikelihood maximised under the constraints of the null is smaller than the unconstrained maximum value. In order to satisfy the null, we need in general only one z in the interior of U such that $F_A(z) = F_B(z)$. Thus, in order to maximise the ELF under the constraint of the null, we begin by computing the maximum where, for a given $z \in Y^\circ$, we impose the condition that $F_A(z) = F_B(z)$. We then choose for the constrained maximum that value of z which gives the greatest value of the constrained ELF.

For given z, the constraint we wish to impose can be written as

$$\sum_{y_t^A \in Y^A} p_t^A \mathrm{I}(y_t^A \leq z) = \sum_{y_s^B \in Y^B} p_s^B \mathrm{I}(y_s^B \leq z), \tag{2}$$

where the $\mathrm{I}(\cdot)$ are indicator functions, equal to 1 if the condition is true, and to 0 if not. If we denote by $F^K(\boldsymbol{p}^K; \cdot)$ the cumulative distribution function with points of support the y_t^K and corresponding probabilities the p_t^K, then it can be seen that condition (2) imposes the requirement that $F^A(\boldsymbol{p}^A, z) = F^B(\boldsymbol{p}^B, z)$.

The maximisation problem can be stated as follows:

$$\max_{p_t^A, p_s^B} \sum_{y_t^A \in Y^A} n_t^A \log p_t^A + \sum_{y_s^B \in Y^B} n_s^B \log p_s^B$$

$$\text{subject to} \quad \sum_{y_t^A \in Y^A} p_t^A = 1, \quad \sum_{y_s^B \in Y^B} p_s^B = 1, \quad \sum_{y_t^A \in Y^A} p_t^A \mathrm{I}(y_t^A \leq z) = \sum_{Y_s^B \in Y^B} p_s^B \mathrm{I}(y_s^B \leq z).$$

The Lagrangian for this constrained maximisation of the ELF is

$$\sum_t n_t^A \log p_t^A + \sum_s n_s^B \log p_s^B + \lambda_A \left(1 - \sum_t p_t^A\right) + \lambda_B \left(1 - \sum_s p_s^B\right)$$
$$-\mu \left(\sum_t p_t^A \mathrm{I}(y_t^A \leq z) - \sum_s p_s^B \mathrm{I}(y_s^B \leq z)\right),$$

with obvious notation for sums over all points in Y^A and Y^B, and where we define Lagrange multipliers λ_A, λ_B, and μ for the three constraints.

The first-order conditions are the constraints themselves and the relations

$$p_t^A = \frac{n_t^A}{\lambda_A + \mu \mathrm{I}(y_t^A \le z)} \quad \text{and} \quad p_s^B = \frac{n_s^B}{\lambda_B - \mu \mathrm{I}(y_s^B \le z)}. \tag{3}$$

Since $\sum_t p_t^A = 1$, we find that

$$\begin{aligned}\lambda_A &= \sum_t \frac{\lambda_A n_t^A}{\lambda_A + \mu \mathrm{I}_t(z)} = \sum_t n_t^A \frac{\lambda_A + \mu \mathrm{I}_t(z)}{\lambda_A + \mu \mathrm{I}_t(z)} - \mu \sum_t \frac{n_t^A \mathrm{I}_t(z)}{\lambda_A + \mu \mathrm{I}_t(z)} \\ &= N_A - \frac{\mu}{\lambda_A + \mu} \sum_t n_t^A \mathrm{I}_t(z) = N_A - \frac{\mu}{\lambda_A + \mu} N_A(z),\end{aligned} \tag{4}$$

where $\mathrm{I}_t(z) \equiv \mathrm{I}(y_t^A \le z)$ and $N_A(z) = \sum_t n_t^A \mathrm{I}_t(z)$ is the number of points in sample A less than or equal to z. Similarly,

$$\lambda_B = N_B + \frac{\mu}{\lambda_B - \mu} N_B(z) \tag{5}$$

with $N_B(z) = \sum_s n_s^B \mathrm{I}_s(z)$. With the relations (3), the constraint (2) becomes

$$\sum_t \frac{\mathrm{I}_t(z)}{\lambda_A + \mu} = \sum_s \frac{\mathrm{I}_s(z)}{\lambda_B - \mu}, \quad \text{that is,} \quad \frac{N_A(z)}{\lambda_A + \mu} = \frac{N_B(z)}{\lambda_B - \mu}. \tag{6}$$

Thus, adding (4) and (5), we see that

$$\lambda_A + \lambda_B = N_A + N_B = N, \tag{7}$$

where $N \equiv N_A + N_B$.

If we make the definition $\nu \equiv \lambda_A + \mu$, then, from (7), $\lambda_B - \mu = N - \lambda_A - \mu = N - \nu$. Thus (6) becomes

$$\frac{N_A(z)}{\nu} = \frac{N_B(z)}{N - \nu}. \tag{8}$$

Solving for ν, we obtain

$$\nu = \frac{N N_A(z)}{N_A(z) + N_B(z)}. \tag{9}$$

From (4), we see that

$$\lambda_A = N_A - N_A(z) + \frac{\lambda_A N_A(z)}{\lambda_A + \mu} \quad \text{so that} \quad 1 = \frac{N_A - N_A(z)}{\lambda_A} + \frac{N_A(z)}{\lambda_A + \mu}. \tag{10}$$

Similarly, from (5),

$$1 = \frac{N_B - N_B(z)}{\lambda_B} + \frac{N_B(z)}{\lambda_B - \mu}. \tag{11}$$

Write $\lambda \equiv \lambda_A$, and define $M_K(z) = N_K - N_K(z)$. Then (10) and (11) combine with (6) to give

$$\frac{M_A(z)}{\lambda} = \frac{M_B(z)}{N-\lambda}. \tag{12}$$

Solving for λ, we see that

$$\lambda = \frac{N M_A(z)}{M_A(z) + M_B(z)}. \tag{13}$$

The probabilities (3) can now be written in terms of the data alone using (9) and (13). We find that

$$p_t^A = \frac{n_t^A \mathrm{I}_t(z)}{\nu} + \frac{n_t^A(1-\mathrm{I}_t(z))}{\lambda} \quad \text{and} \quad p_s^B = \frac{n_s^B \mathrm{I}_s(z)}{N-\nu} + \frac{n_s^B(1-\mathrm{I}_s(z))}{N-\lambda}. \tag{14}$$

We may use these in order to express the value of the ELF maximised under constraint as

$$\sum_t n_t^A \log n_t^A + \sum_s n_s^B \log n_s^B$$
$$-N_A(z)\log\nu - M_A(z)\log\lambda - N_B(z)\log(N-\nu) - M_B(z)\log(N-\lambda). \tag{15}$$

Twice the difference between the unconstrained maximum (1), which can be written as

$$\sum_t n_t^A \log n_t^A + \sum_s n_s^B \log n_s^B - N_A \log N_A - N_B \log N_B,$$

and the constrained maximum (15) is an empirical likelihood ratio statistic, which of course does not have the conventional χ^2 distribution under the null hypothesis that the constraint (2) is true in the populations.

Using (9) and (13) for ν and λ, the statistic can be seen to satisfy the relation

$$\begin{aligned} \tfrac{1}{2}\mathrm{LR(z)} = {} & N\log N - N_A \log N_A - N_B \log N_B + N_A(z)\log N_A(z) + N_B(z)\log N_B(z) \\ & + M_A(z)\log M_A(z) + M_B(z)\log M_B(z) - \big(N_A(z)+N_B(z)\big)\log\big(N_A(z)+N_B(z)\big) \\ & - \big(M_A(z)+M_B(z)\big)\log\big(M_A(z)+M_B(z)\big). \end{aligned} \tag{16}$$

We will see later how to use the statistic in order to test the hypothesis of nondominance.

3. The Minimum t Statistic

In Kaur, Prakasa Rao, and Singh (1994), a test is proposed based on the minimum of the t statistic for the hypothesis that $F_A(z) - F_B(z) = 0$, computed for each value of z in some closed interval contained in the interior of U. The minimum value is used as the test statistic for the null of nondominance against the alternative of dominance. The test can be interpreted as an intersection-union test. It is shown that the probability of rejection of the null when it is true is asymptotically bounded by the nominal level of a test based on the standard normal distribution. Howes (1993) proposed a very similar intersection-union test, except that the t statistics are calculated only for the predetermined grid of points.

In this section, we show that the empirical likelihood ratio statistic (16) developed in the previous section, where the constraint is that $F_A(z) = F_B(z)$ for some z in the interior of U, is locally equivalent to the square of the t statistic with that constraint as its null, under the assumption that indeed $F_A(z) = F_B(z)$.

Since we have assumed that the two samples are independent, the variance of $\hat{F}_A(z) - \hat{F}_B(z)$ is just the sum of the variances of the two terms. The variance of $\hat{F}_K(z)$, $K = A, B$, is $F_K(z)\big(1 - F_K(z)\big)/N_K$, where N_K is as usual the size of the sample from population K, and this variance can be estimated by replacing $F_K(z)$ by $\hat{F}_K(z)$. Thus the square of the t statistic is

$$t^2(z) = \frac{N_A N_B \big(\hat{F}_A(z) - \hat{F}_B(z)\big)^2}{N_B \hat{F}_A(z)\big(1 - \hat{F}_A(z)\big) + N_A \hat{F}_B(z)\big(1 - \hat{F}_B(z)\big)}. \tag{17}$$

Suppose that $F_A(z) = F_B(z)$ and denote their common value by $F(z)$. Also define $\varDelta(z) \equiv \hat{F}_A(z) - \hat{F}_B(z)$. For the purposes of asymptotic theory, we consider the limit in which, as $N \to \infty$, N_A/N tends to a constant r, $0 < r < 1$. It follows that $\hat{F}_K(z) = F(z) + O_p(N^{-1/2})$ and that $\varDelta(z) = O_p(N^{-1/2})$ as $N \to \infty$.

The statistic (17) can therefore be expressed as the sum of its leading-order asymptotic term and a term that tends to 0 as $N \to \infty$:

$$t^2(z) = \frac{r(1-r)}{F(z)(1 - F(z))} \operatorname*{plim}_{N\to\infty} N\varDelta^2(z) + O_p(N^{-1/2}). \tag{18}$$

It can now be shown that the statistic $\mathrm{LR}(z)$ given by (16) is also equal to the right-hand side of (18) under the same assumptions as those that led to (18). The algebra is rather messy, and so we state the result as a theorem.

Theorem 1

> As the size N of the combined sample tends to infinity in such a way that $N_A/N \to r$, $0 < r < 1$, the statistic $\mathrm{LR}(z)$ defined by (16) tends to the right-hand side of (18) for any point z in the interior of U, the union of the supports of populations A and B, such that $F_A(z) = F_B(z)$.

Proof: In Appendix. ■

Remarks:

It is important to note that, for the result of the above theorem and for (18) to hold, the point z must be in the *interior* of U. As we will see in the next section, the behaviour of the statistics in the tails of the distributions is not adequately represented by the asymptotic analysis of this section.

It is clear that both of the two statistics are invariant under monotonically increasing transformations of the measurement units, in the sense that if an income z is transformed into an income z' in a new system of measurement, then $t^2(z)$ in the old system is equal to $t(z')$ in the new, and similarly for $\mathrm{LR}(z)$.

Corollary

> Under local alternatives to the null hypothesis that $F_A(z) = F_B(z)$, where $F_A(z) - F_B(z)$ is of order $N^{-1/2}$ as $N \to \infty$, the local equivalence of $t^2(z)$ and $\mathrm{LR}(z)$ continues to hold.

Proof:

Let $F_A(z) = F(z)$ and $F_B(z) = F(z) - N^{-1/2}\delta(z)$, where $\delta(z)$ is independent of N. Then $\Delta(z)$ is still of order $N^{-1/2}$ and the limiting expression on the right-hand side of (18) is unchanged. The common asymptotic distribution of the two statistics now has a positive noncentrality parameter. ■

4. The Tails of the Distribution

Although the null of nondominance has the attractive property that, if it is rejected, all that is left is dominance, this property comes with a caveat. The caveat is that it is impossible to infer dominance over the full support of the distributions if these distributions are continuous in the tails. This is also a warning that non-rejection of the literature's earlier null hypotheses of dominance cannot be interpreted as implying dominance. Moreover and as we shall see in this section, compared to these tests of dominance, the tests of nondominance that we consider have the advantage of delimiting the range over which restricted dominance can be inferred.

The nondominance of distribution A by B can be expressed by the relation

$$\max_{z \in U} F_B(z) - F_A(z) \geq 0, \tag{19}$$

where U is as usual the joint support of the two distributions. But if z_- denotes the lower limit of U, we must have $F_B(z_-) - F_A(z_-) = 0$, whether or not the null is true. Thus the maximum over the whole of U is never less than 0. Rejecting (19) by a statistical test is therefore impossible. The maximum may well be significantly greater than 0, but it can never be significantly less, as would be required for a rejection of the null.

Of course, an actual test is carried out, not over all of U, but only at the elements of the set Y of points observed in one or other sample. Suppose that A is dominated by B in the sample. Then the smallest element of Y is the smallest observation, y_1^A, in the sample drawn from A. The squared t statistic for the hypothesis that $F_A(y_1^A) - F_B(y_1^A) = 0$ is then

$$t_1^2 \equiv \frac{N_A N_B (\hat{F}_A^1 - \hat{F}_B^1)^2}{N_B \hat{F}_A^1 (1 - \hat{F}_A^1) + N_A \hat{F}_B^1 (1 - \hat{F}_B^1)},$$

where $\hat{F}_K^1 = \hat{F}_K(y_1^A)$, $K = A, B$; recall (17). Now $\hat{F}_B^1 = 0$ and $\hat{F}_A^1 = 1/N_A$, so that

$$t_1^2 = \frac{N_A N_B / N_A^2}{(N_B/N_A)(1 - 1/N_A)} = \frac{N_A}{N_A - 1}.$$

The t statistic itself is thus approximately equal to $1 + 1/(2N_A)$. Since the minimum over Y of the t statistics is no greater than t_1, and since $1 + 1/(2N_A)$ is nowhere near the critical value of the standard normal distribution for any conventional significance level, it follows that rejection of the null of nondominance is impossible. A similar, more complicated, calculation can be performed for the test based on the empirical likelihood ratio, with the same conclusion.

If the data are discrete, discretised or censored in the tails, then it is no longer impossible to reject the null if there is enough probability mass in the atoms at either end or over the censored areas of the distribution. If the distributions are continuous but are discretised or censored, then it becomes steadily more difficult to reject the null as the discretisation becomes finer, and in the limit once more impossible. The difficulty is clearly that, in the tails of continuous distributions, the amount of information conveyed by the sample tends to zero, and so it becomes impossible to discriminate among different hypotheses about what is going on there. Focussing on *restricted* stochastic dominance is then the only empirically sensible avenue.

5. Restricted stochastic dominance and distributional rankings

Interestingly, there does exist in welfare economics and in finance an empirical tradition of considering restricted dominance. One reason for this is the suspicion formalised below that testing for unrestricted dominance is too statistically demanding since it forces comparisons of dominance curves over areas that effectively make use of too little information. A second reason is the feeling that unrestricted dominance does not impose sufficient limits on the ranges over which certain ethical principles must be obeyed.

It is often argued for instance that the precise value of the living standards of those that are abjectly deprived should not be of concern to empirical researchers: the number of such abjectly deprived people should be sufficient information for welfare analysts. It does not matter for social evaluation purposes what the exact value of one's income is when it is clearly too low. In other words, the distribution of living standards under some low threshold should not matter: everyone under that threshold should certainly be deemed to be in very difficult circumstances. Such views militate in favour of the use of *restricted* poverty indices, indices that give intuitively equal ethical weight to all those who are below a survival poverty line. The same views also suggest an analogous concept of restricted social welfare.

To see this more precisely, consider the case in which we are interested in whether there is more poverty in a distribution A than in a distribution B. To establish this, consider for expositional simplicity the case of additive poverty indices, denoted as $P_A(z)$ for a distribution A:

$$P_A(z) = \int \pi(y;z)dF_A(y) \tag{20}$$

where z is a poverty line, y is income, $F_A(s)$ is the cumulative distribution function for distribution A, and $\pi(y;z) \geq 0$ is the poverty contribution to total poverty of someone with income y, with $\pi(y;z) = 0$ whenever $y > z$. This definition is general enough to encompass many of the poverty indices that are used in the empirical literature. Also assume that $\pi(y;z)$ is differentiable in y between 0 and z, let its first-order derivative with respect to y be given by $\pi_1(y;z)$, and let $Z = [z^-, z^+]$, with z^- and z^+ being respectively some lower and upper limits to the range of possible poverty lines. Then denote by $\Pi^1(Z)$ the class of "first-order" poverty indices $P(z)$ defined in terms of a function $\pi(y;z)$ that satisfies the condition

$$\pi_1(y;z) \begin{cases} \leq 0 & \text{if } y \in Z, \\ = 0 & \text{otherwise;} \end{cases} \quad \text{for all } z \in Z. \tag{21}$$

We are interested in checking whether $\Delta P(z) \equiv P_A(z) - P_B(z) \geq 0$ for all such poverty indices. This can be done using the following statement of first-order poverty dominance:

(First-order poverty dominance)

$$\Delta P(z) > 0 \text{ for all } P(z) \in \Pi^1(Z) \text{ iff } \Delta F(y) > 0 \text{ for all } y \in Z, \tag{22}$$

with $\Delta F(y) \equiv F_A(y) - F_B(y)$. Note that (22) is reminiscent of the *restricted headcount* ordering of Atkinson (1987). Unlike Atkinson's result, however, the ordering in (22) is valid for an entire class $\Pi^1(Z)$ of indices. Traditional unrestricted poverty dominance is obtained with $Z = [0, z^+]$.

The indices that are members of $\Pi^1(Z)$ are insensitive to changes in incomes when these take place outside of Z: they thus behave like the headcount index outside Z. This avoids being concerned with the precise living standards of the most deprived – for some, a possibly controversial ethical procedure, but unavoidable from a statistical and empirical point of view. To illustrate this, let the poverty gap at y be defined as $g(y;z) = \max(z-y,0)$. For a distribution A, the popular FGT indices are then given (in their un-normalised form) by:

$$P_A(z;\alpha) = \int g(y;z)^{\alpha} dF_A(y)$$

for $\alpha \geq 0$. One example of a headcount-like restricted index that is ordered by (22) is then given by:

$$P(z) = \begin{cases} F(z^-) & \text{when } z \in [0, z^-], \\ F(z) & \text{when } z \in [z^-, z^+], \\ F(z^+) & \text{when } z \geq z^+. \end{cases} \tag{23}$$

The formulation in (23) can be justified by a view that a poverty line cannot sensibly lie below z^-: anyone with z^- or less should necessarily be considered as being in equally abject deprivation. Another example of a poverty index that is ordered by (22) is:

$$P(z) = \begin{cases} zF(z^-) & \text{when } z < z^-; \\ zF(z^-) + \int_{F(z^-)}^{F(z^+)} g(y;z)dF(y) & \text{when } z \geq z^-. \end{cases} \tag{24}$$

$P(z)$ in (24) is the same as the traditional average poverty gap when all incomes below z^- are lowered to 0, again presumably because everyone with z^- or less is deemed to be in abject deprivation. When $z \geq z^-$, the index in (24) then reacts similarly to the poverty headcount for incomes below z^-, since changing (marginally) the value of these incomes does not change the index. For higher incomes (up to z^+), (24) behaves as the traditional average poverty gap.

A setup for restricted social welfare dominance can proceed analogously, *e.g.* by using utilitarian functions defined as

$$W = \int u(y)dF(y),$$

and by allowing $u(y)$ to be strictly monotonically increasing only over some restricted range of income Z. Verifying whether $\Delta F(y) > 0$ for all $y \in Z$ is then the test for restricted first-order welfare dominance. Fixing $Z = [0, \infty[$ yields traditional first-order welfare dominance.

6. Testing restricted dominance

A natural way to proceed, in cases in which there is dominance in the sample, is then to seek an interval $[\hat{z}^-, \hat{z}^+]$ over which one *can* reject the hypothesis

$$\max_{z \in [\hat{z}^-, \hat{z}^+]} F_B(z) - F_A(z) \geq 0. \tag{25}$$

For simplicity, we concentrate in what follows on the lower bound $\hat{z}^-$.

As the notation indicates, $\hat{z}^-$ is random, being estimated from the sample. In fact, it is useful to conceive of $\hat{z}^-$ in much the same way as the limit of a confidence interval. We consider a nested set of null hypotheses, parametrised by z^-, of the form

$$\max_{z \in [z^-, z^+]} F_B(z) - F_A(z) \geq 0, \tag{26}$$

where the upper limit z^+ is fixed in such a way that the maximum of $F_B(z) - F_A(z)$ is comfortably greater than $F_B(z^+) - F_A(z^+)$. As z^- increases, the hypothesis becomes progressively more constrained, and therefore easier to reject. For some prespecified nominal level α, one then defines $\hat{z}^-$ as the smallest value of z^- for which the hypothesis (26) can be rejected at level α by the chosen test procedure, which could be based either on the minimum t statistic or the minimised empirical likelihood ratio. It is possible that $\hat{z}^- = z^+$, in which case none of the nested set of null hypotheses can be rejected at level α. With this definition, $\hat{z}^-$ is analogous to the upper limit β_+ of a confidence interval for some parameter β. Just as $\hat{z}^-$ is the smallest value of z^- for which (26) can be rejected, so β_+ is the smallest value of β_0 for which the hypothesis $\beta = \beta_0$ can be rejected at (nominal) level α, where $1 - \alpha$ is the desired confidence level for the interval.

The analogy can be pushed a little further. The length of a confidence interval is related to the power of the test on which the confidence interval is based. Similarly, $\hat{z}^-$ is related to the power of the test of nondominance. The closer is $\hat{z}^-$ to the bottom of the joint support of the distributions, the more powerful is our rejection of nondominance. Thus a study of the statistical properties of $\hat{z}^-$ is akin to a study of the power of a conventional statistical test.

7. Testing the Hypothesis of Nondominance

We have at our disposal two test statistics to test the null hypothesis that distribution B does not dominate distribution A, the two being locally equivalent in some circumstances. In what follows, we assume that, if the distributions are continuous, they are discretised in the tails, so as to allow for the possibility that the null hypothesis may be rejected. Empirical distribution functions (EDFs) are computed for the two samples, after discretisation if necessary, and evaluated at all of the points y_t^A and y_s^B of the samples. It is convenient to suppose that both samples have been sorted in increasing order, so that $y_t^A \leq y_{t'}^A$ for $t < t'$. The EDF for sample A, which we denote by $\hat{F}_A(\cdot)$, is of course constant on each interval of the form $[y_t^A, y_{t+1}^A[$, and a similar result holds for the EDF of sample B, denoted $\hat{F}_B(\cdot)$.

Recall that we denote by Y the set of all the y_t^A, $t = 1, \ldots, N_A$, and the y_s^B, $s = 1, \ldots, N_B$. If $\hat{F}_B(y) < \hat{F}_A(y)$ for all $y \in Y$ except for the largest value of y_s^B, then we say that B dominates A in the sample. The point $y_{N_B}^B$ is excluded from Y because, with dominance in the sample, it is the largest value observed in the pooled sample, and so $\hat{F}_A(y_{N_B}^B) = \hat{F}_B(y_{N_B}^B) = 1$. On the other hand, we do not exclude the smallest value y_1^A, since $\hat{F}_A(y_1^A) = n_1^A/N_A$ while $\hat{F}_B(y_1^A) = 0$. Obviously, it is only when there is dominance in the sample that there is any possible reason to reject the null of nondominance.

When there is dominance in the sample, let us redefine the set Y° to be Y without the upper end-point $Y_{N_B}^B$ only. Then the minimum t statistic of which the square is given by (17) can be found by minimising $t(z)$ over $z \in Y^\circ$. There is no loss of generality in restricting the search for the maximising z to the elements of Y°, since the quantities $N_K(z)$ and $M_K(z)$ on which (15) depends are constant on the intervals between elements of Y° that are adjacent when the elements are sorted. Thus the element $\hat{z} \in Y^\circ$ which maximises (15) can be found by a simple search over the elements of Y°.

Since the EDFs are the distributions defined by the probabilities that solve the problem of the unconstrained maximisation of the empirical loglikelihood function, they define the unconstrained maximum of that function. For the empirical likelihood test statistic, we also require the maximum of the ELF constrained by the requirement of nondominance. This constrained maximum is given by the ELF (15) for the value $\tilde{z}$ that maximises (15). Again, $\tilde{z}$ can be found by search over the elements of Y°.

The constrained empirical likelihood estimates of the CDFs of the two distributions can be written as

$$\tilde{F}_K(z) = \sum_{y_t^K \le z} p_t^K n_t^K,$$

where the probabilities p_t^K are given by (14) with $z = \tilde{z}$. Normally, $\tilde{z}$ is the only point in Y° for which $\tilde{F}_A(z)$ and $\tilde{F}_B(z)$ are equal. Certainly, there can be no z for which $\tilde{F}_A(z) < \tilde{F}_B(z)$ with strict inequality, since, if there were, the value of ELF could be increased by imposing $\tilde{F}_A(z) = \tilde{F}_B(z)$, so that we would have $\text{ELF}(z) > \text{ELF}(\tilde{z})$, contrary to our assumption. Thus the distributions $\tilde{F}_A$ and $\tilde{F}_B$ are on the frontier of the null hypothesis of nondominance, and they represent those distributions contained in the null hypothesis that are closest to the unrestricted EDFs, for which there is dominance, by the criterion of the empirical likelihood.

For the remainder of our discussion, we restrict the null hypothesis to the frontier of nondominance, that is, to distributions such that $F_A(z_0) = F_B(z_0)$ for exactly one point z_0 in the interior of the joint support U, and $F_A(z) > F_B(z)$ with strict inequality for all $z \neq z_0$ in the interior of U. These distributions constitute the least favourable case of the hypothesis of nondominance in the sense that, with either the minimum t statistic or the minimum EL statistic, the probability of rejection of the null is no smaller on the frontier than with any other configuration of nondominance. This result follows from the following theorem.

Theorem 2

Suppose that the distribution F_B is changed so that the new distribution weakly stochastically dominates the old at first order. Then, for any z in the interior of the joint support U, the new distribution of the statistic $t(z)$ of which the square is given by (17) and the sign by that of $\hat{F}_A(z) - \hat{F}_B(z)$ weakly stochastically dominates its old distribution at first order. The same is true for the square root of the statistic $\text{LR}(z)$ given by (16) signed in the same way. If F_A is changed so that the old distribution weakly stochastically dominates the new at first order, the same conclusions hold.

Proof: In Appendix. ■

Remarks:

The changes in the statement of the theorem all tend to move the distributions in the direction of greater dominance of A by B. Thus we expect that they lead to increased probabilities of rejection of the null of nondominance. If, as the theorem states, the new distributions of the test statistics dominate the old, that means that their right-hand tails contain more probability mass, and so they indeed lead to higher rejection probabilities.

We are now ready to state the most useful consequence of restricting the null hypothesis to the frontier of nondominance.

Theorem 3

The minima over z of both the signed asymptotic t statistic $t(z)$ and the signed empirical likelihood ratio statistic $\text{LR}^{1/2}(z)$ are asymptotically pivotal for the null hypothesis that the distributions A and B lie on the frontier of nondominance of A by B, that is, there exists exactly one z_0 in the interior of the joint support U of the two distributions for which $F_A(z_0) = F_B(z_0)$, while $F_A(z) > F_B(z)$ strictly for all $z \neq z_0$ in the interior of U.

Proof: In Appendix. ■

Remarks:

Theorem 3 shows that we have at our disposal two test statistics suitable for testing the null hypothesis that distribution B does not dominate distribution A stochastically at first order, namely the minima of $t(z)$ and $\text{LR}^{1/2}(z)$. For configurations that lie on the frontier of this hypothesis, as defined above, the asymptotic distribution of both statistics is $\text{N}(0, 1)$. By Theorem 2, use of the quantiles of this distribution as critical values for the test leads to an asymptotically conservative test when there is nondominance inside the frontier.

It is clear from the remark following the proof of Theorem 1 that both statistics are invariant under monotonic transformations of the measuring units of income.

The fact that the statistics are asymptotically pivotal means that we can use the bootstrap to perform tests that should benefit from asymptotic refinements in finite samples; see Beran (1988). We study this possibility by means of simulation experiments in the next section.

8. Simulation Experiments

There are various things that we wish to vary in the simulation experiments discussed in this section. First is sample size. Second is the extent to which observations are discretised in the tails of the distribution. Third is the way in which the two populations are configured. In those experiments in which we study the rejection probability of various tests under the null, we wish most of the time to have population A dominated by population B except at one point, where the CDFs of the two distributions are equal, When we wish to investigate the power of the tests, we allow B to dominate A to a greater or lesser extent.

Stochastic dominance to first order is invariant under increasing transformations of the variable z that is the argument of the CDFs F_A and F_B. It is therefore without loss of generality that we define our distributions on the $[0, 1]$ interval. We always let population A be uniformly distributed on this interval: $F_A(z) = z$ for $z \in [0, 1]$. For population B, the interval is split up into eight equal segments, with the CDF being linear on each segment. In the base configuration, the cumulative probabilities at the upper limit of each segment are 0.03, 0.13, 0.20, 0.50, 0.57, 0.67, 0.70, and 1.00. This is contrasted with the evenly increasing cumulative probabilities for A, which are 0.125, 0.25, 0.375, 0.50, 0.625, 0.75, 0.875, and 1.00. Clearly B dominates A everywhere except for $z = 0.5$, where $F_A(0.5) = F_B(0.5) = 0.5$. This base configuration is thus on the frontier of the null hypothesis of nondominance, as discussed in the previous section. In addition, we agglomerate the segments $[0, 0.1]$ and $[0.9, 1]$, putting the full probability mass of the segment on $z = 0.1$ and $z = 0.9$ respectively.

In Table 1, we give the rejection probabilities of two asymptotic tests, based on the minimised values of $t(z)$ and $\text{LR}^{1/2}(z)$, as a function of sample size. The samples drawn from A are of sizes N_A = 16, 32, 64, 128, 256, 512, 1024, 2048, and 4096. The corresponding samples from B are of sizes N_B = 7, 19, 43, 81, 187, 379, 763, 1531, and 3067, the rule being $N_B = 0.75N_A - 5$. The results are based on 10,000 replications. Preliminary experiments showed that, when the samples from the two populations were of the same size, or of sizes with a large greatest common divisor, the possible values of the statistics were so restricted that their distributions were lumpy. For our purposes, this lumpiness conceals more than it reveals, and so it seemed preferable to choose sample sizes that were relatively prime.

The two test statistics turn out to be very close indeed in value when each is minimised over z. This is evident in Table 1, but the results there concern only the tail of the distributions of the statistics. In Figure 1, we graph P value plots for the two statistics, over the full range from 0 to 1. See Davidson and MacKinnon (1998) for a discussion of P value plots, in which is plotted the CDF of the P value for the test.

Two sample sizes are shown: $N_A = 32$ and $N_A = 256$. In the latter case, it is hard to see any difference between the plots for the two statistics, and even for the much smaller sample size, the differences are plainly very minor indeed.

In the experimental setup that gave rise to Figure 1, it was possible to cover the full range of the statistics, since, even when there was nondominance in the sample,

Table 1

N_A	$\alpha = 0.01$ $t_{\min}$	$\alpha = 0.01$ $\mathrm{LR}_{\min}$	$\alpha = 0.05$ $t_{\min}$	$\alpha = 0.05$ $\mathrm{LR}_{\min}$	$\alpha = 0.10$ $t_{\min}$	$\alpha = 0.10$ $\mathrm{LR}_{\min}$
16	0.001	0.000	0.005	0.005	0.013	0.017
32	0.000	0.000	0.004	0.004	0.017	0.015
64	0.001	0.001	0.009	0.010	0.026	0.030
128	0.003	0.003	0.021	0.021	0.048	0.047
256	0.001	0.006	0.033	0.033	0.070	0.069
512	0.010	0.010	0.039	0.039	0.082	0.082
1024	0.007	0.007	0.042	0.042	0.087	0.087
2048	0.010	0.010	0.043	0.043	0.087	0.087
4096	0.009	0.009	0.044	0.044	0.092	0.092

Rejection probabilities, asymptotic tests, base case, α = nominal level

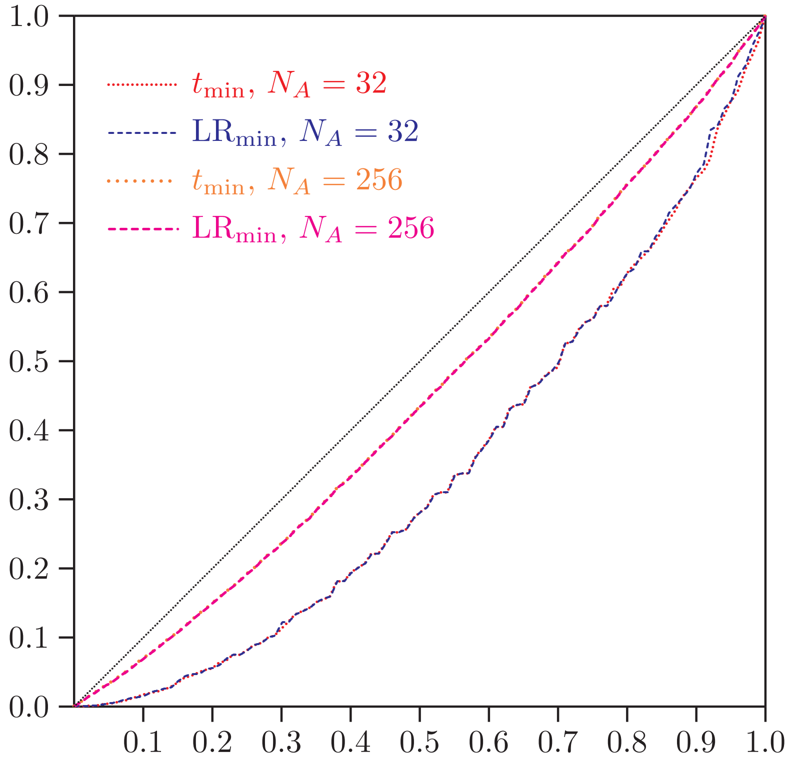

Figure 1: *P* value plots for asymptotic tests

we could evaluate the statistics as usual, obtaining negative values. This was for illustrative purposes only. In practice, one would stop as soon as nondominance is observed in the sample, thereby failing to reject the null hypothesis.

It is clear from both Table 1 and Figure 1 that the asymptotic tests have a tendency to underreject, a tendency which disappears only slowly as the sample sizes grow larger. This is hardly surprising. If the point of contact of the two distributions is at $z = z_0$, then the distribution of $t(z_0)$ and $\text{LR}^{1/2}(z_0)$ is approximately standard normal. But minimising with respect to z always yields a statistic that is no greater than those evaluated at z_0. Thus the rejection probability can be expected to be smaller, as we observe.

We now consider bootstrap tests based on the minimised statistics. In bootstrapping, it is essential that the bootstrap samples are generated by a bootstrap data-generating process (DGP) that satisfies the null hypothesis, since we wish to use the bootstrap in order to obtain an estimate of the distribution of the statistic being bootstrapped *under the null hypothesis*. Here, our rather artificial null is the frontier of nondominance, on which the statistics we are using are asymptotically pivotal, by Theorem 3.

Since the results we have obtained so far show that the two statistics are very similar even in very small samples, we may well be led to favour the minimum t statistic on the basis of its relative simplicity. But the procedure by which the empirical likelihood ratio statistic is computed also provides a very straightforward way to set up a suitable bootstrap DGP. Once the minimising z is found, the probabilities (14) are evaluated at that z, and these, associated with the realised sample values, the y_t^A and the y_s^B, provide distributions from which bootstrap samples can be drawn.

The bootstrap DGP therefore uses discrete populations, with atoms at the observed values in the two samples. In this, it is like the bootstrap DGP of a typical resampling bootstrap. But, as in Brown and Newey (2002), the probabilities of resampling any particular observation are not equal, but are adjusted, by maximisation of the ELF, so as to satisfy the null hypothesis under test. In our experiments, we used bootstrap DGPs determined in this way using the probabilities (14), and generated bootstrap samples from them. Each of these is automatically discretised in the tails, since the "populations" from which they are drawn have atoms in the tails. For each bootstrap sample, then, we compute the minimum statistics just as with the original data. Bootstrap P values are then computed as the proportion of the bootstrap statistics that are greater in value than the statistic from the original data.

In Table 2, we give results like those in Table 1, but for bootstrap tests rather than asymptotic tests. For each replication, 399 bootstrap statistics were computed, Results are given only for the empirical likelihood statistic, since the t statistic gave results that were indistinguishable.

It is not necessary, and it would have taken a good deal of computing time, to give results for sample sizes greater than those shown, since the rejection probabilities are not significantly different from nominal already for $N_A = 128$.

In Figure 2, P value plots are given for $N_A = 32$ and 128, for the asymptotic and bootstrap tests based on the empirical likelihood statistic. This time, we show results only for P values less than 0.5, since the bootstrap DGP is not appropriate in cases of nondominance in the sample.

Table 2

N_A	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
32	0.001	0.018	0.051
64	0.003	0.033	0.082
128	0.007	0.046	0.100
256	0.011	0.054	0.104
512	0.011	0.049	0.100
1024	0.011	0.053	0.105

Rejection probabilities, bootstrap tests, base case, α = nominal level

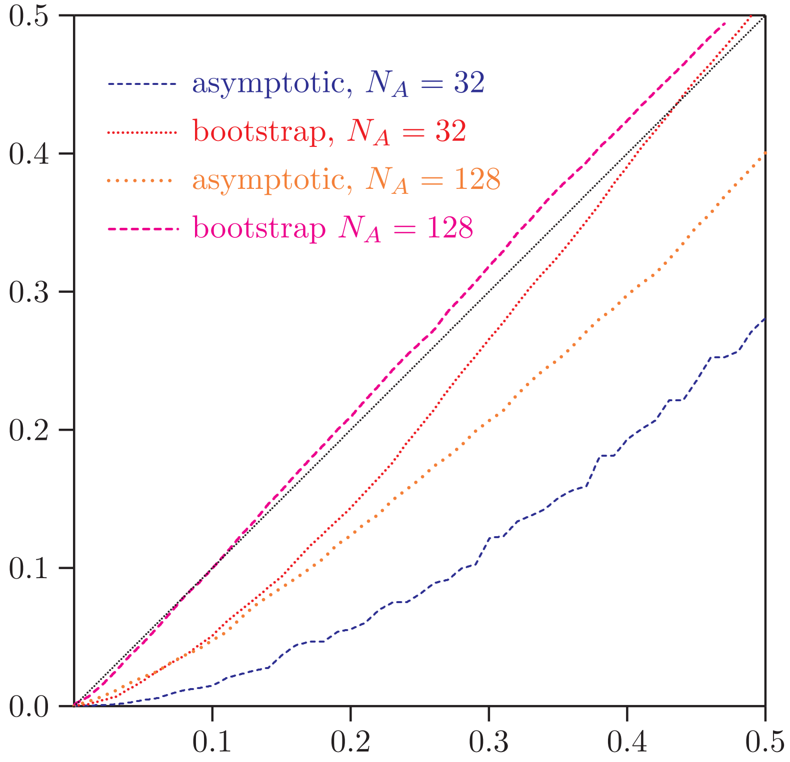

Figure 2: *P* value plots for asymptotic and bootstrap tests

In the bootstrap context, if there is nondominance in the original samples, no bootstrapping is done, and a P value of 1 is assigned. If there is dominance in the original samples, an event which under the null has a probability that tends to one half as the sample sizes tend to infinity, then bootstrapping is undertaken; each time the bootstrap generates a pair of samples without dominance, since the bootstrap test statistic would be negative, and so not greater than the positive statistic from the original samples, this bootstrap replication does not contribute to the P value. Thus a bootstrap DGP that generates many samples without dominance leads to small P values and frequent rejection of the null of nondominance.

From the figure, we see that, like the asymptotic tests, the bootstrap test suffers from a tendency to underreject in small samples. However, this tendency disappears much more quickly than with the asymptotic tests. Once sample sizes are around 100, the bootstrap seems to provide very reliable inference.

We now look at the effects of altering the amount of discretisation in the tails of the distributions. It turns out that these effects are quite different for the asymptotic and bootstrap tests. In Figure 3 are shown P value plots for the asymptotic test for the base case with $N_A = 128$, for different amounts of agglomeration, the same amount in each tail. It can be seen that the rejection rate diminishes steadily as the discretisation is performed for values of z_- progressively further into the tails, where discretisation occurs for $z < z_-$ and for $z > 1 - z_-$. This behaviour is entirely as expected, in accord with the discussion in Section 4. For values of z_- in the range 0.10 to 0.16, the P value plots are essentially identical.

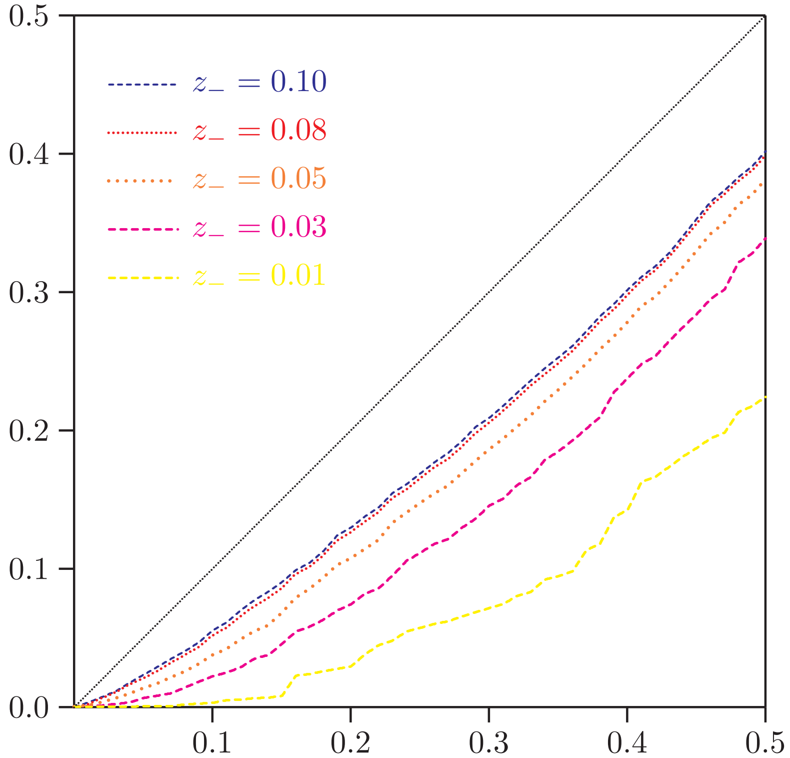

Figure 3: Effects of varying amounts of discretisation in the tails

With the bootstrap, the same is true for $z_- \in [0.07, 0, 13]$, but for either smaller or greater values of z_- the bootstrap test rejects the null on almost every occasion when there is dominance in the sample. For small values of z_-, the bootstrap DGP usually makes the CDFs of the two bootstrap populations touch somewhere in one of the tails rather in the middle of the distribution. This means that the bootstrap DGP is very different from the true DGP, in such a way that there is almost always nondominance in the bootstrap samples. Thus the bootstrap P value, as the proportion of bootstrap statistics greater than the original one, is very close to zero.

The same sort of thing happens with $z_- > 0.14$. It appears that the excessive agglomeration again leads to a bootstrap DGP where the populations touch at one of the tail atoms, with similar consequences. Whereas with small values of z_-, the overrejection is probably a generic phenomenon, it may be an artefact of our particular base case for larger values of z_-.

The base case we have considered so far is one in which B dominates A substantially except at one point in the middle of the distribution. We now consider two other configurations, the first in which the two distributions still touch in the middle, but the dominance by B is less elsewhere. The cumulative probabilities at the upper limits of the eight segments in this case are 0.1, 0.2, 0.3, 0.5, 0.6, 0.7, 0.8, and 1.0. The second configuration has the two distributions touching twice, for values of z equal to 0.25 and 0.75. The cumulative probabilities are 0.10, 0.25, 0.35, 0.45, 0.55, 0.75, 0.85, and 1.00. With z_- set to 0.1, the tests are now all conservative, with rejection probabilities well below nominal in reasonably small samples. Figure 4 shows the P value plots for $N_A = 64$ and $N_B = 43$ with tests based on the minimum t statistic. As usual, the empirical likelihood statistic gives essentially indistinguishable results.

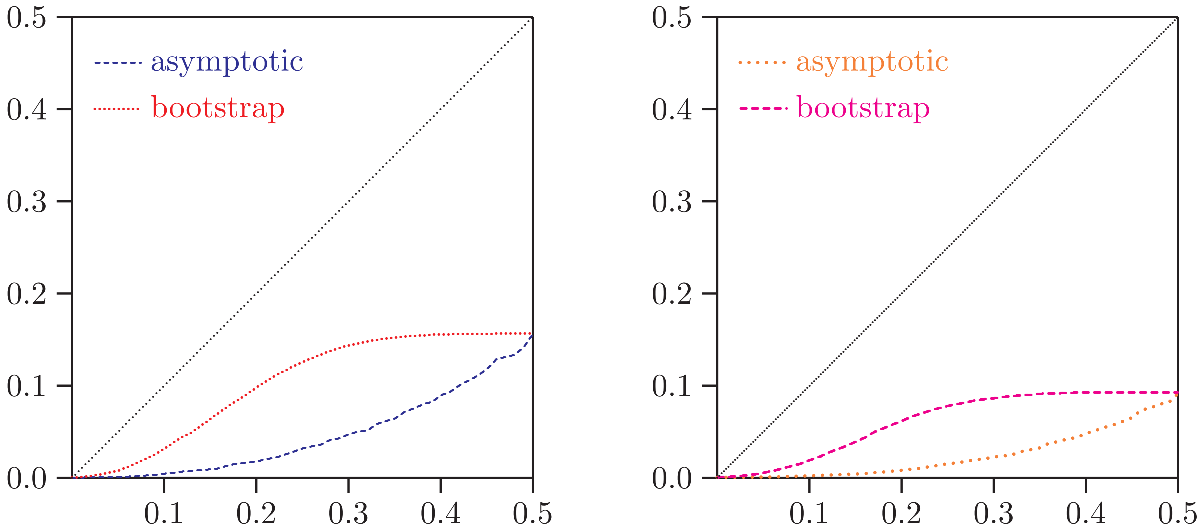

Figure 4: Alternative configurations: first on left, second on right

In the second configuration, in which the distributions touch twice, the tests are still more conservative than in the first configuration. In both cases, it can be seen that the bootstrap test is a good deal less conservative than the asymptotic one. This fact means also that the P value plots for the bootstrap test flatten out more quickly, corresponding to cases in which there is nondominance in the samples. It is also apparent that, in these two configurations, the probability of dominance in the original data, which is the asymptote to which the P value plots tend, is substantially less than a half.

Another configuration that we looked at needs no graphical presentation. If both populations correspond to the uniform distribution on $[0, 1]$, rejection of the null of nondominance simply did not occur in all our replications. Of course, when the distributions coincide over their whole range, we are far removed from the frontier of the null hypothesis, and so we expect to have conservative tests.

We now turn our attention to considerations of power. Again, we study two configurations in which population B dominates A. In the first, we modify our base configuration slightly, using as cumulative probabilities at the upper limits of the segments the values 0.03, 0.13, 0.20, 0.40, 0.47, 0.57, 0,70, and 1.00. There is therefore clear dominance in the middle of the distribution. The second configuration uses cumulative probabilities of 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, and 1.0. This distribution is uniform until the last segment, which has a much greater probability mass than the others.

In Figure 5, various results are given, with those for the first configuration in the left-hand panel and the second in the two right-hand panels. Both asymptotic and bootstrap tests based on the minimum t statistic are considered, and z_- is set to 0.1. There is nothing at all surprising in the left-hand panel. We saw in Figure 2 that, with the base configuration, the asymptotic test underrejects severely for $N_A = 32$ and $N_B = 19$. Here, the rejection rate is still less than nominal level for those sample sizes. With the base configuration, the bootstrap test also underrejects, but less severely, and here it achieves a rejection rate modestly greater than the significance level. For $N_A = 64$ and $N_B = 43$, the increased power brought by larger samples is manifest. The asymptotic test gives rejection rates modestly greater than the level, but the bootstrap test does much better, with a rejection rate of slightly more than 14% at a 5% level, and nearly 28% at a 10% level.

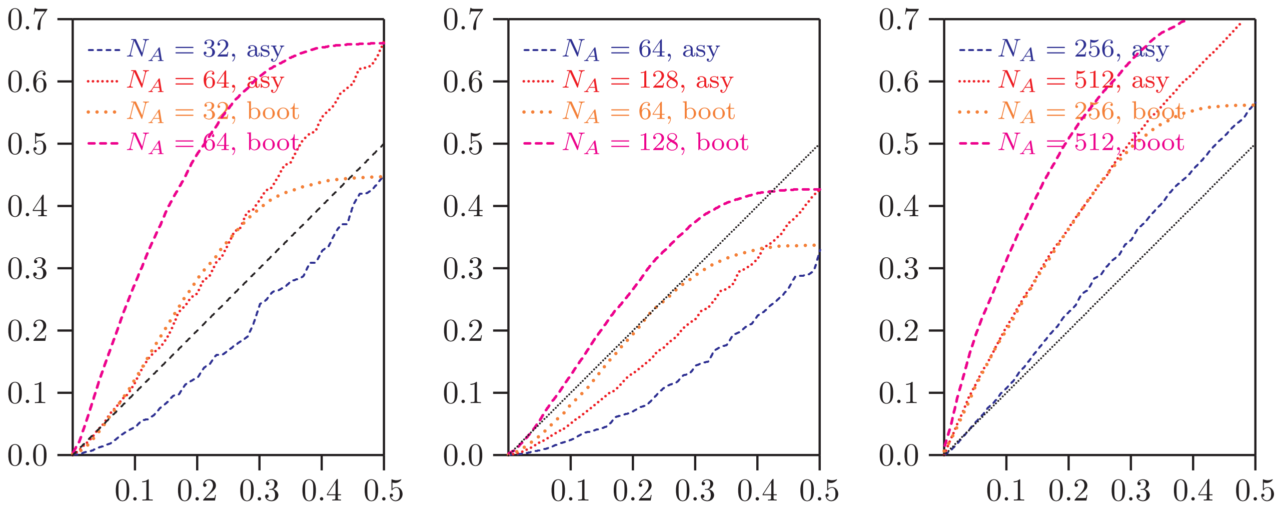

Figure 5: Power curves

In the second configuration, power is uniformly much less. If we were to change things so that the null of nondominance was satisfied, say by increasing the cumulative probability in population B for z around 0.25, then the results shown in Figure 4 indicate that the tests would be distinctly conservative. Here we see the expected counterpart when only a modest degree of dominance is introduced, namely low power. Even for $N_A = 128$, the rejection rate of the asymptotic test is always smaller than the significance level. With the larger sample sizes of the right-hand panel, some ability to reject is seen, but it is not at all striking with $N_A = 256$. In contrast, the bootstrap test has some power for all sample sizes except $N_A = 64$, and its rejection rate rises rapidly in larger samples, although rejection rates comparable to those obtained with the first configuration with $N_A = 64$ are attained only for N_A somewhere between 256 and 512.

The possible configurations of the two populations are very diverse indeed, and so the results presented here can only be indicative. However, a pattern that emerges consistently is that bootstrap tests outperform their asymptotic counterparts in terms of both size and power. They are less subject to the severe underrejection displayed by asymptotic tests even when the configuration is on the frontier of the null hypothesis, and they provide substantially better power to reject the null when it is significantly false.

Conventional practice often discretises data, transforming them so that the distributions have atoms at the points of a grid. Essentially, the resulting data are sampled from discrete distributions. A few simulations were run for such data. The results were not markedly different from those obtained for continuous data, discretised only in the tails. The tendency of the asymptotic tests to underreject is slightly less marked, because the discretisation means that the minimising z is equal to the true (discrete) z_0 with high probability. However, the lumpiness observed when the two sample sizes have a large greatest common divisor is very evident indeed, and prevents simulation results from being as informative as those obtained from continuous distributions.

9. Discussion and Conclusions

In this paper, we have adopted the point of view that, if we really wish to demonstrate statistically that the distribution of population B stochastically dominates that of population A at first order, then it is appropriate to use a null hypothesis of nondominance, since, if we reject it, all that is left is dominance. However, we show that is it impossible to reject this null at any conventional significance level if we have continuous distributions and use all the observations in samples drawn from them. With discrete distributions, this problem does not necessarily arise, and indeed, in practice, many investigators explicitly or implicitly discretise their samples by setting up a grid of points and agglomerating observations in the samples on to atoms at the points of the grid.

If we are ready in the case of continuous distributions to discretise in the tails of the distributions at least, then we have seen that it is easy to set up both asymptotic and bootstrap tests for the null of nondominance. We consider two seemingly different statistics, one the minimum t statistic of KPS, the other an empirical likelihood ratio statistic. We show that the two statistics typically take on very similar values in practice, and that inference using one of them is indistinguishable from inference using the other. The advantage of the empirical likelihood ratio statistic is that, in order to compute it, we compute a set of probabilities that estimate the probabilities of the populations under the hypothesis that they are at the frontier of nondominance, that is, that they are such that there is dominance of A by B everywhere except at exactly one point in the interior of the common support of the distributions.

This fact makes it possible to use the bootstrap in order to estimate the distributions of either one of the two statistics under data-generating processes that are on the frontier of nondominance. In fact, we show that the statistics are asymptotically pivotal on the frontier, so that we can expect that the bootstrap will provide more reliable inference than the asymptotic distributions of the statistics. This turns out to be the case in a selection of configurations that we study by means of simulation experiments. Our preferred testing procedure is thus a bootstrap procedure, in which the bootstrap samples are generated using the probabilities computed in the process of evaluating the empirical likelihood ratio statistic. It does not seem to matter whether the minimum t statistic or the likelihood ratio statistic is used.

Most of the literature on testing relations between a pair of distributions deals with tests for which the null hypothesis is dominance. It is plausible to suppose that these tests too can be dealt with by the methods of empirical likelihood, but it is less simple to do so. For this sort of test, we do not reject the null of dominance unless there is nondominance in the sample. In that case, we wish to find the distributions that respect the null of dominance and are closest, by the criterion of the empirical likelihood, to the unrestricted estimates that exhibit nondominance. These distributions must of course lie on the frontier of the null hypothesis. In general, however, it is not enough to require that there should be just one point $y \in Y$ at which the restricted estimates coincide. In Wolak (1989), this matter is considered for the case of discrete distributions, and it is shown that locating the pair of distributions closest to a pair of sample distributions which display nondominance involves the solution of a quadratic programming problem. Further, the asymptotic distribution of the natural test statistic, under a DGP lying on the frontier, is a mixture of chi-squared distributions that is not as simple to treat as the standard normal asymptotic distributions found in this paper. It remains for future research to see whether empirical likelihood methods, used with continuous distributions, can simplify tests with a null of dominance.

Appendix

Proof of Theorem 1:

For $K = A, B$, $N_K(z) = N_K\hat{F}_K(z)$ and $M_K(z) = N_K(1-\hat{F}_K(z))$. Therefore

$$\begin{aligned} &N_K(z)\log N_K(z) + M_K(z)\log M_K(z) \\ =&N_K\log N_K + N_K\big(\hat{F}_K(z)\log\hat{F}_K(z) + (1-\hat{F}_K(z))\log(1-\hat{F}_K(z))\big). \end{aligned} \tag{27}$$

Further,

$$\Big(\sum_{K=A,B} N_K(z)\Big)\log\Big(\sum_{K=A,B} N_K(z)\Big) + \Big(\sum_{K=A,B} M_K(z)\Big)\log\Big(\sum_{K=A,B} M_K(z)\Big) =$$

$$N\log N + \Big(\sum_{K=A,B} N_K\hat{F}_K(z)\Big)\log\Big(\sum_{K=A,B}\frac{N_K}{N}\hat{F}_K(z)\Big) +$$

$$\Big(\sum_{K=A,B} N_K(1-\hat{F}_K(z))\Big)\log\Big(\sum_{K=A,B}\frac{N_K}{N}(1-\hat{F}_K(z))\Big) \qquad (28)$$

From (16), we see that $\mathrm{LR}(z)$ is equal to twice the expression

$$\sum_{K=A,B}\Big(N_K(z)\log N_K(z) + M_K(z)\log M_K(z) - N_K\log N_K\Big) + N\log N$$

$$-\Big(\sum_{K=A,B} N_K(z)\Big)\log\Big(\sum_{K=A,B} N_K(z)\Big) + \Big(\sum_{K=A,B} M_K(z)\Big)\log\Big(\sum_{K=A,B} M_K(z)\Big)$$

From (27) and (28), this expression can be written as

$$-\sum_{K=A,B} N_K\hat{F}_K(z)\log\Big(\frac{N_A\hat{F}_A(z) + N_B\hat{F}_B(z)}{N\hat{F}_K(z)}\Big)$$

$$-\sum_{K=A,B} N_K(1-\hat{F}_K(z))\log\Big(\frac{N-(N_A\hat{F}_A(z) + N_B\hat{F}_B(z))}{N(1-\hat{F}_K(z))}\Big). \qquad (29)$$

Consider now the first sum in the above expression, which can be written as

$$-\big(N_A\hat{F}_A(z) + N_B\hat{F}_B(z)\big)\log\big(N_A\hat{F}_A(z) + N_B\hat{F}_B(z)\big)$$

$$+N_A\hat{F}_A(z)\log N\hat{F}_A(z) + N_B\hat{F}_B(z)\log N\hat{F}_B(z). \qquad (30)$$

Define $\mathit{\Delta}(z) \equiv \hat{F}_A(z) - \hat{F}_B(z)$. Then we see that $N_A\hat{F}_A(z) + N_B\hat{F}_B(z) = N\hat{F}_B(z) + N_A\mathit{\Delta}(z)$. Making these substitutions lets us write expression (30) as

$$-\big(N\hat{F}_B(z) + N_A\mathit{\Delta}(z)\big)\Big(\log N\hat{F}_B(z) + \log\Big(1 + \frac{N_A\mathit{\Delta}(z)}{N\hat{F}_B(z)}\Big)\Big)$$

$$+N_A(\hat{F}_B(z) + \mathit{\Delta}(z))\big(\log N\hat{F}_B(z) + \log\big(1 + \mathit{\Delta}(z)/\hat{F}_B(z)\big)\big) + N_B\hat{F}_B(z)\log N\hat{F}_B(z).$$

Taylor expanding up to second order in $\mathit{\Delta}(z)$ then gives

$$(-N + N_A + N_B)\hat{F}_B(z)\log N\hat{F}_B(z) - N_A\mathit{\Delta}(z) + \tfrac{1}{2}\frac{N_A^2\mathit{\Delta}^2(z)}{N\hat{F}_B(z)} - N_A\mathit{\Delta}(z)\log N\hat{F}_B(z)$$

$$-\frac{N_A^2\mathit{\Delta}^2(z)}{N\hat{F}_B(z)} + N_A\mathit{\Delta}(z) - \tfrac{1}{2}\frac{N_A\mathit{\Delta}^2(z)}{\hat{F}_B(z)} + N_A\mathit{\Delta}(z)\log N\hat{F}_B(z) + \frac{N_A\mathit{\Delta}^2(z)}{\hat{F}_B(z)} + O_p(N^{-1/2}),$$

since, under our assumptions, $N_K = O_p(N)$ and $\Delta(z) = O_P(N^{-1/2})$. The term independent of $\Delta(z)$ in the above expression and the terms linear in $\Delta(z)$ all cancel, and so what remains is just a term of order unity and a remainder that tends to zero as $N \to \infty$:

$$\frac{1}{2}\frac{N_A(N-N_A)\Delta^2(z)}{N\hat{F}_B(z)} + O_p(N^{-1/2}) = \frac{1}{2}\frac{N_A N_B \Delta^2(z)}{N\hat{F}_B(z)} + O_p(N^{-1/2}).$$

Since $\hat{F}_B(z) = F(z) + O_p(N^{-1/2})$, this expression is equal to $N_A N_B \Delta^2(z)/2NF(z)$ to the same order. An exactly similar calculation for the second line of (29) shows that, to the same order of approximation, it is equal to $N_A N_B \Delta^2(z)/2N(1-F(z))$. The entire expression (29) is therefore

$$\frac{1}{2}\frac{N_A N_B \Delta^2(z)}{N}\Big(\frac{1}{F(z)} + \frac{1}{1-F(z)}\Big) = \frac{1}{2}\frac{N_A N_B \Delta^2(z)}{NF(z)(1-F(z))} + O_p(N^{-1/2}). \qquad (31)$$

Finally, since $N_A/N \to r$ as $N \to \infty$ and $N_B/N \to 1-r$, we see that the large-sample limit of $\mathrm{LR}(z)$, which is twice that of (31), is

$$\frac{r(1-r)}{F(z)(1-F(z))} \operatorname*{plim}_{N\to\infty} N\Delta^2(z),$$

which is the leading-order term on the right-hand side of (18), as required. ■

Proof of Theorem 2:

The proof relies on the following construction, which is trivial if we are dealing with two continuous distributions, but requires some care if either one or both contain atoms.

Consider two CDFs F and G such that G weakly stochastically dominates F at first order. Let Y be a random variable of which the CDF is F. We construct a random variable Z, of which the CDF is G, as follows. If y is a realisation of Y, the corresponding realisation of Z is computed according to a rule that depends on whether y is a continuity point of F. If it is, let $p = F(y)$. Then z, the realisation of Z, is given by

$$z = \sup_w \{w \,|\, G(w) < p\}. \qquad (32)$$

Thus, if there exists a w such that $G(w) = p$, then, since G is continuous to the right, we have

$$z = \inf_w \{w \,|\, G(w) = p\}. \qquad (33)$$

Otherwise, if there is no w with $G(w) = p$, there must exist w such that G is discontinuous at w, jumping from a value $p_- < p$ to the left of w to a value $p_+ > p$ at w. Then it follows from (32) that $z = w$.

Next suppose that y is an atom of the distribution F, such that $F(y-) = p_-$ and $F(y) = p_+$. Let $p = p_- + (p_+ - p_-)u$, where u is a realisation of a random variable distributed uniformly on the $[0, 1]$ interval, independently of anything else. Then z is again computed using the formula (32) with this new definition of p.

Denote by P the random variable of which the realisation is p, defined as a function of the realisation y and possibly also a realisation u. Then P is distributed uniformly on $[0, 1]$. To see this, first let p be such that there exists y such that $p = F(y)$ and y is a continuity point of F. Then $\Pr(P \leq p) = \Pr(F(Y) \leq F(y)) = \Pr(Y \leq y) = F(y) = p$. Next, let p be such that F has an atom at some y for which $F(y-) = p_- < p$ and $F(y) = p_+ \geq p$. The event $P \leq p$ can then be realised in two ways: first, if $Y < y$, of which the probability is p_-, and, second, if $Y = y$ and the realisation u satisfies the condition $p_- + (p_+ - p_-)u \leq p$. Since Y and the random variable of which u is a realisation are independent, the probability of this second event is the product of the probability that $Y = y$, which is $p_+ - p_-$, and $(p - p_-)/(p_+ - p_-)$, the probability that the realisation u satisfies the required condition. Adding up, we find that, in this case,

$$\Pr(P \leq p) = p_- + (p_+ - p_-)\frac{p - p_-}{p_+ - p_-} = p.$$

Thus in all cases $\Pr(P \leq p) = p$, so that P is distributed uniformly on $[0, 1]$.

We now have to show that Z, defined in terms of P by means of the relation (32) between their realisations, is such that $Z \geq Y$ almost surely, and that it has G for its CDF. Note first that (32) defines z as a (weakly) increasing function of p. Now suppose on the one hand that p is such that there exists w with $G(w) = p$, Then, by (33), the realisation z is the smallest such w. It follows that $\Pr(Z \leq w) = \Pr(P \leq p) = p = G(z)$. Further, since $G(z) = p \geq F(y-) \geq G(y-)$ given that G dominates F, it follows that $z \geq y$ unless $G(z) = F(z) = G(y-) = F(y-)$ and $y > z$. However, in this last case, realisations of Y in the range $[z, y[$ are of probability zero, so that the set of realisations with $y > z$ has probability zero. If on the other hand there is no w with $G(w) = p$, then, as above,there must exist a w at which G has an atom, with $G(w-) = p_-$ and $G(w) = p_+$; $p \in\,]p_-, p_+[$, and $z = w$. But then $\Pr(Z \leq w) = \Pr(P \leq p_+) = G(w)$. In this case, we have that $G(w) > p \geq F(y-) \geq G(y-)$, and so again $w \geq y$. Thus in all cases $w \geq y$, so that $Z \geq Y$ almost surely, and $\Pr(Z \leq w) = G(w)$, as we wished to show.

Each random sample of independent observations from the distribution F can be converted into a random sample from the distribution G by transforming each realisation y into the realisation z by the above construction. Let $\mathcal{Y}^N$ be the random variable realisations of which are IID random samples drawn from F. We may also define the random variable $\mathcal{Z}^N$ as a deterministic function of $\mathcal{Y}^N$, by converting the sample element by element. The random variable $\mathcal{Z}^N$ then has the same distribution as the random variable realisations of which are IID random samples drawn from G. Since every observation from F is smaller than the corresponding one in the converted sample, it follows that the EDF of the sample from F is stochastically dominated at first order by that from G almost surely.

Applying this result to a change in distribution F_B to a new distribution that stochastically dominates the old one, we see that, for any random sample drawn from F_B, and for all z, the EDF $\hat{F}_B(z)$ of the converted sample becomes smaller. Similarly, a change in F_A towards a more dominated distribution makes the EDF $\hat{F}_A(z)$ greater. Further, the minimum value of $t(z)$ or of $\mathrm{LR}(z)$ after such changes cannot be less than before. Thus the theorem is proved once we show that the square root statistics defined in the statement of the theorem are increasing functions of $\hat{F}_A(z)$ for all z and decreasing functions of $\hat{F}_B(z)$.

We compute the derivative with respect to $\hat{F}_A(z)$ of $t(z)$ as given by the square root of expression (17). This square root can be written in the form

$$C\frac{x-y}{\big(x(1-x)+k\big)^{1/2}} \tag{34}$$

where $x = \hat{F}_A(z)$, $y = \hat{F}_B(z)$, $k = (N_A/N_B)\hat{F}_B(z)(1-\hat{F}_B(z))$, and C is a positive constant. The derivative of expression (34) with respect to x is C times

$$\frac{2x(1-x)+2k-(x-y)(1-2x)}{2\big(x(1-x)+k\big)^{3/2}}.$$

This expression is certainly positive unless $x-y$ and $1-2x$ have the same sign. Suppose first that $x \le 1/2$ and $x-y>0$. Then, since $y \ge 0$, $x \ge x-y$ and so

$$2x(1-x)-(x-y)(1-2x) \ge 2x(1-x)-x(1-2x) = x \ge 0.$$

Similarly, if $x \ge 1/2$ and $x-y<0$, we see that $|x-y| \le 1-x$. Then

$$2x(1-x)-(y-x)(2x-1) \ge 2x(1-x)-(1-x)(2x-1) = 1-x \ge 0.$$

Thus the derivative is positive in all cases. The proof that the derivative of $t(z)$ with respect to $\hat{F}_B(z)$ is negative is exactly similar.

The statistic $\mathrm{LR}(z)$ is given by twice the expression (29). The first line of (29) is in turn equal to (30), of which the derivative with respect to $\hat{F}_A(z)$ is

$$\begin{aligned}-N_A\log\big(N_A\hat{F}_A(z)+N_B\hat{F}_B(z)\big)-N_A+N_A\log N\hat{F}_A(z)+(N_A/N)N\\ = -N_A\log\Big(1-\frac{N_B\varDelta(z)}{N\hat{F}_A(z)}\Big).\end{aligned}$$

Since $N_B/(N\hat{F}_A(z))$ is positive, this expression has the same sign as $\varDelta(z)$. Similarly, the derivative of the second line of (29) with respect to $\hat{F}_A(z)$ is

$$N_A\log\Big(1+\frac{N_B\varDelta(z)}{N(1-\hat{F}_A(z))}\Big),$$

of which the sign is also the same as that of $\varDelta(z)$. Since the square root statistic is defined to have the same sign as $\varDelta(z)$, its derivative with respect to $\hat{F}_A(z)$ is everywhere nonnegative. This completes the proof. ■

Proof of Theorem 3:

Under the restricted null hypothesis of the statement of the theorem, the statistic $t(z_0)$ is distributed asymptotically as $\mathrm{N}(0,1)$. The probability that $t(z_0) \leq z_{1-\alpha}$, where $z_{1-\alpha}$ is the $(1-\alpha)$ quantile of $\mathrm{N}(0,1)$, therefore tends to $1-\alpha$ as $N \to \infty$. The probability that the minimum over $z \in Y^\circ$ of $t(z)$ is less than $z_{1-\alpha}$ is therefore no smaller than $1-\alpha$ asymptotically. Thus the probability of rejecting the null of nondominance on the basis of the minimum of $t(z)$ is no greater than α. This is the standard intersection-union argument used to justify the use of the minimum of $t(z)$ as a test statistic.

In Theorem 2.2 of KPS, it is shown that, if the distributions A and B belong to the restricted null hypothesis, then the probability of rejecting the null is actually equal to α asymptotically. We conclude therefore that the asymptotic distribution of the minimum of $t(z)$ is $\mathrm{N}(0,1)$. Since this is a unique distribution, it follows that this statistic is asymptotically pivotal for the restricted null. The local equivalence of $t(z)$ and $\mathrm{LR}^{1/2}(z)$ shown in Theorem 1 then extends the result to the empirical likelihood ratio statistic. ■

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