Finite and Large Sample Inference for One- and Two-Factor Stochastic Volatility Models *

Jean-Marie Dufour[†] Pascale Valéry[‡]

9th May 2005

*We are grateful to Eric Jacquier, Lynda Khalaf and Nour Meddahi for helpful comments.
†CIREQ,CIRANO,Université de Montreal,email: jean.marie.dufour@umontreal.ca
‡HEC-Montréal,IFM^2,email: pascale.valery@hec.ca

ABSTRACT

In this paper, we study the problem of testing hypotheses on the parameters of a stochastic volatility model (SV), allowing for the possible presence of identification problems that can lead to nonstandard asymptotic distributional theory. We focus on the development of exact procedures whose size can be controlled in finite samples, as well as computationally simple large-sample tests with better finite-sample properties. We consider Wald-type, score-type and likelihood-ratio-type tests based on a computationally simple moment (non maximum-likelihood) estimator. We also propose a $C(\alpha)$-type test which is very easy to implement and exhibits relatively good size and power properties. Besides usual linear restrictions on the SV model coefficients, the problems studied include testing homoskedasticity against a SV alternative – which raises identification issues - as well testing a specification test for the null hypothesis of linear volatility against a fractionally integrated model on volatility. Based on the above testing methodology, we further test for the null hypothesis of two factors driving the dynamic of the volatility process against only one factor. Three ways of implementing tests based on alternative statistics are compared: asymptotic critical values (when available), a local Monte Carlo test (or parametric bootstrap) procedure, and a maximized Monte Carlo (MMC) procedure. The size and power properties of the proposed procedures are examined in a simulation experiment. The results indicate that the $C(\alpha)$-based tests have the best size and power properties, while Monte Carlo tests are much more reliable than those based on asymptotic critical values. Further, in cases where the parametric bootstrap appears to fail (for example, in the presence of identification problems), the MMC procedure easily controls the level of the tests. Finally, we present an application to a long time series of returns on the Standard and Poor's Composite Price Index.

Key words: exact tests; Monte Carlo tests; $C(\alpha)$-tests; specification test; stochastic volatility; fractionally integrated volatility; two-factor volatility process.
JEL classification: C1, C13, C12, C32, C15.

Contents

List of Definitions, Propositions and Theorems

1. Introduction

Evaluating the likelihood function of ARCH models is relatively easy compared to Stochastic Volatility models (SV) for which it is impossible to get an explicit closed-form expression for the likelihood function [see Shephard (1996), Mahieu and Schotman (1998)]. This is a generic feature common to almost all nonlinear latent variable models due to the curse of the high dimensionality of the integral appearing in the likelihood function of the stochastic volatility model. This is the reason why econometricians were reluctant to use this kind of models in their applications for a long time since in this setting, maximum likelihood methods are computationally intensive. But ever since progress has been made regarding the estimation of nonlinear latent variable models in general and stochastic volatility models in particular. It mainly exists three types of methods, namely, quasi-exact methods, simulation-based-estimation methods and bayesian methods. Thus, we can mention the Quasi Maximum Likelihood (QML) approach suggested by Nelson (1988) and Harvey, Ruiz and Shephard (1994), Ruiz (1994), a Generalized Method of Moments(GMM) procedure proposed by Melino and Turnbull (1990). On the other hand, increased computer power has made simulation-based estimation methods more attractive among which we can mention the Simulated Method of Moments (SMM) proposed by Duffie and Singleton (1993), the indirect inference approach of Gouriéroux, Monfort and Renault (1993) and the moment matching methods of Gallant and Tauchen (1996). But computer intensive Markov Chain Monte Carlo methods applied to SV models by Jacquier, Polson and Rossi (1994) and Kim and Shephard (1994), Kim, Shephard and Chib (1998), Wong(2002a,2002b) and simulation-based Maximum Likelihood (SML) method proposed by Danielsson and Richard (1993), Danielsson (1994), are the most efficient methods to estimate this kind of models. In particular, Danielsson (1994), Danielsson and Richard (1993) develop an importance sampling technique to estimate the integral appearing in the likelihood function of the SV model. In a Bayesian setting, Jacquier, Polson and Rossi (1994), Kim, Shephard and Chib (1998) combine a Gibbs sampler with the Metropolis-Hastings algorithm to obtain the marginal posterior densities of the parameters of the SV model.

In contrast, the major contribution of this paper is to provide asymptotic and exact inference techniques for testing hypotheses on the parameters of the log-normal stochastic volatility model with an autoregressive mean part. Indeed, the standard form as set forth, for instance, in Harvey, Ruiz, and Shephard (1994), Jacquier, Polson, and Rossi (1994), Danielsson (1994), takes the form of an autoregression whose innovations are scaled by an unobservable volatility process, usually distributed as a lognormal autoregression but other distributions (Student, mixture of normal distributions) can be considered [see Kim, Shephard and Chib (1998), Mahieu and Schotman (1998), Wong (2002a,2002b)]. The stochastic volatility specification we have chosen here comes from Gallant, Hsieh, Tauchen (1997), Tauchen (1996). Whereas all the authors quoted above, mainly focus on estimation performance for the stochastic volatility model, often preoccupied by efficiency considerations [e.g. bayesian methods, Efficient Method of Moments], our paper instead is mostly motivated by inference techniques applied to the stochastic volatility model. Our concern for inference, in particular for simulation-based inference such as the technique of Monte Carlo tests introduced by Dwass (1957) for permutation tests, and later extended by Barnard (1963) and Birnbaum (1974), requires an estimation method easy to implement. Thus, the estimation method used in this paper is mainly a method of moments [see Taylor (1986)] in two steps which coincides with the GMM procedure in the particular case that the autoregressive mean part vanishes. For a detailed presentation of the estimation technique applied to the SV model with an autoregressive conditional mean part, see Dufour and Valéry (2004). As econometricians previously quoted mainly focused on efficient estimation procedures to estimate the SV model, they mostly examined specification tests such as the χ^2 tests for goodness of fit in Andersen and Sorensen (1996), Andersen, Chung and Sorensen (1999), specification tests with diagnostics in Gallant, Hsieh and Tauchen (1997), χ^2 specification tests through Indirect Inference criterion in Monfardini (1997), or likelihood ratio tests statistics for comparative fit in Kim, Shephard and Chib (1998). As a result, inference techniques for testing hypotheses on parameters of the stochastic volatility model remained underdeveloped, apart from standard t-tests for individual parameters in Andersen and Sorensen (1996), in Andersen, Chung and Sorensen (1999) often performed with size distorsions.

In this setting, the aim of the paper is to fulfill the gap for testing hypotheses on parameters of the SV model, more precisely, to propose exact tests in the sense that the tests have correct levels in small samples. To do this, we implement the three standard test statistics that is the Wald-type, score-type and likelihood-ratio-type test based on a computationally simple method-of-moments estimator available in closed form [see Dufour and Valéry (2004)]. We further consider a $C(\alpha)$-type test [see Ronchetti (1987), Berger and Wallenstein (1989), Kocherlakota and Kocherlakota (1991)] which is very easy to implement in our framework and demonstrates good size and power properties. Using these test procedures, we test the null hypothesis of no persistence in the volatility against alternatives of strong persistence in the volatility process.Testing for the presence or not of strong serial correlation in the volatility process is relevant mostly for speculative returns which tend to display systematic long-range volatility dependencies in general and more specifically for option pricing predictions. Indeed, a strong serial correlation in the underlying volatility process will help minimizing the pricing error of future option prices computed on the basis of both current realized and implied volatilities. In this respect, a stream of the option pricing literature has seized the importance of this issue by allowing for long-range dependence in the volatility process when compared with the standard stochastic volatility model or the ARCH family, using thereby a fractional integration process whose autocorrelation function is known to decrease at a much slower rate, a hyperbolic decay rate, than that of the standard stochastic volatility process or the ARCH-type family [see Breidt,Crato, Lima (1998) for detection and estimation of a long-memory feature in a discrete time stochastic volatility model , see Comte and Renault (1998) for the continuous time stochastic volatility and Comte, Coutin and Renault (2003), Ohanissian, Russel and Tsay (2003) for its applications to option pricing]. In this regards, we propose a specification test for testing the null hypothesis of linearity in the volatility process against a fractionally integrated volatility process by means of a likelihood-ratio-type test statistic for comparative fit. Furthermore,we also provide a joint test for testing homoskedasticity in the volatility process. In this respect, a statistical check for homoskedasticity in the stochastic volatility model could be viewed as a relevant pre-test before trying to include a latent factor to drive the dynamic of the volatility process which makes its estimation much more complicated. We further test for the null hypothesis of one factor driving the dynamic of the volatility process against two SV factors. As pointed out by Chernov *et al* (2003), Durham (2004a, 2004b), standard SV models are not able to capture all aspects of asset returns distribution such as *tail thickness*. As a solution, introducing a second factor in the volatility dynamic may act as a factor dedicated to exclusive modelling of tail behavior while the first factor may rather model the long memory component. Eraker *et al*(2003) propose to model the same feature by introducing a jump component to the SV factor. However, testing for homoskedasticity arises strong anomalies as the moment conditions become no more identifying under the null. Similarly, when testing for the number of factors in the volatility process, the correlation parameter between the two factors become unidentified under the null arising thereby severe identification issues. In presence of such irregularities, the standard asymptotic distribution is known to fail and one has to resort to either nonstandard inference techniques or simulation-based inference techniques such as Monte Carlo tests to still conduct valid inference in such situations. Andrews (2001) derives the asymptotic null and local alternative distributions of quasi-likelihood ratio, rescaled quasi-likelihood ratio, Wald, and score tests when standard regularity conditions fail to hold. As an example, he tests the null hypothesis of no conditional heteroskedasticity in a GARCH(1,1) regression model which makes the GARCH AR parameter unidentified under the null.

In a Monte Carlo study we compare the finite sample properties of the standard asymptotic techniques to the technique of Monte Carlo tests which is valid in finite samples and allow for test statistics whose null distribution may depend on nuisance parameters. In particular maximized Monte Carlo tests (MMC) introduced by Dufour (2002) have the exact level in finite samples when the p-value function is maximized over the entire set of nuisance parameters. In contrast to MMC tests which are highly computer intensive, simplified (asymptotically justified) approximate versions of Monte Carlo tests provide a halfway solution which achieves to control the level of the tests while being less computationally demanding. We finally illustrate the test procedures by providing an application on a long time return series on the Standard and Poor's Composite Price Index.

The paper is organized as follows. Section 2 sets the framework and the assumptions underlying the one-factor SV model and reviews the estimation procedure used to implement the tests. Section 3 presents the two-factor SV model along with the corresponding moment conditions used in the estimation procedure. Section 4 is devoted to the specification test of linear volatility against fractionally integrated volatility in the one-factor SV model. Hypothesis testing is examined in Section 5 where we also discuss how to build confidence sets by inverting the test statistics. In Section 6 we review the technique of Monte Carlo tests. Simulation results are displayed in Section 7 while empirical results on the Standard and Poor's Composite Price Index 500 return series are discussed in Section 8. Section 9 finally concludes.

2. The one-factor SV model

The basic form of the stochastic volatility model we study here for y_t comes from Gallant, Hsieh, Tauchen (1997). Let y_t denote the first difference over a short time interval, a day for instance, of the log-price of a financial asset traded on securities markets.

Assumption 2.1 *The process $\{y_t,\ t \in \mathbb{N}\}$ follows a stochastic volatility model of the type:*

$$y_t - \mu_y = \sum_{j=1}^{L_y} c_j (y_{t-j} - \mu_y) + exp(w_t/2) r_y z_t \ , \tag{2.1}$$

$$w_t - \mu_w = \sum_{j=1}^{L_w} a_j (w_{t-j} - \mu_w) + r_w v_t \ , \tag{2.2}$$

where μ_y, $\{c_j\}_{j=1}^{L_y}$, r_y, μ_w, $\{a_j\}_{j=1}^{L_w}$ and r_w are the parameters of the two equations, called the mean and volatility equations respectively. $s_t = (y_t, w_t)'$ is initialized from its stationary distribution.

The lag lengths of the autoregressive specifications used in the literature are typically short, e.g. $L_w = 1$, $L_y = 1$, or $L_y = 0$, or $L_w = 2$, $L_y = 2$ [see e.g. Andersen and Sorensen (1996), Gallant, Hsieh, Tauchen (1997), Andersen, Chung and Sorensen (1999)]. In this regards, a simplified version of model (2.1)-(2.2) consists in setting $\mu_w = 0$ and $c_j = a_j = 0$, $\forall j \geq 2$, and $\rho = (c, \theta_1')'$ with $\theta_1 = (a, r_y, r_w)'$. We then have:

$$y_t - \mu_y = c(y_{t-1} - \mu_y) + \exp(w_t/2) r_y z_t \ , \quad |c| < 1 \tag{2.3}$$

$$w_t = a w_{t-1} + r_w v_t \ , \quad |a| < 1 \ . \tag{2.4}$$

We shall call the model represented by equations (2.3)-(2.4) the stochastic volatility model with an autoregressive mean part of order one [AR(1)-SV for short].

Assumption 2.2 *The vectors $(z_t, v_t)'$, $t \in \mathbb{N}$ are i.i.d. according to a $N(0, I_2)$ distribution.*

Assumption 2.3 *The process $s_t = (y_t, w_t)'$ is strictly stationary.*

The process is Markovian of order $L_s = max(L_y, L_w)$. Let

$$\rho = (\mu_y, c_1, \ldots, c_{L_y}, r_y, \mu_w, a_1, \ldots, a_{L_w}, r_w)' \tag{2.5}$$

denote the parameter vector of the stochastic volatility model. The process $\{y_t\}$ is observed whereas $\{w_t\}$ is regarded as latent. Accordingly, the joint density of the vector of observations $\bar{y} = (y_1, \ldots, y_T)$ is not available in closed form since it requires evaluating an integral with dimension equal to the whole path of the latent volatilities. Let $F(\bar{y}) = F(y_1, \ldots, y_T) = P[Y_1 \leq y_1, \ldots, Y_T \leq y_T|\rho]$ denote its unknown distribution function

To estimate the AR(1)-SV model above, we consider a two-step method whose first step consists in applying ordinary least squares (OLS) to the mean equation which yields a consistent estimate of the autoregressive parameter c and of the mean parameter μ_y, denoted by $\hat{c}_T$, $\hat{\mu}_{yT}$ and the residuals $\hat{u}_t \equiv u_t(\hat{c}_T) = y_t - \mu_y - \hat{c}_T(y_{t-1} - \mu_y)$. Then, we apply in a second step a method of moments to the residuals $\hat{u}_t$ to get the estimate of the parameter $\theta_1 = (a, r_y, r_w)'$ of the mean and volatility equations. In the sequel we will focus on the particular case where $\mu_y = 0$ but all the results still hold in the general case. In the two propositions below, we recall the moments of the volatility process as well as the estimating equations defining the moment estimator of θ. For a detailed proof of these propositions, the reader is referred to Dufour and Valéry (2004).

Proposition 2.4 MOMENTS OF THE VOLATILITY PROCESS.
Under Assumptions **2.1**,**2.2**,**2.3**, *with* $\mu_y = \mu_w = 0$ *and* $c_j = a_j = 0,\ \forall\ j \geq 2$. *Then* u_t *has the following moments for even values of* k *and* l*:*

$$\mu_k(\theta_1) \equiv E(u_t^k) = r_y^k \frac{k!}{2^{(k/2)}(k/2)!} \exp[\frac{k^2}{8} r_w^2/(1-a^2)] \ , \tag{2.6}$$

$$\begin{aligned} \mu_{k,l}(m|\theta_1) &\equiv E(u_t^k u_{t+m}^l) \\ &= r_y^{k+l} \frac{k!}{2^{(k/2)}(k/2)!} \frac{l!}{2^{(l/2)}(l/2)!} \exp[\frac{r_w^2}{8(1-a^2)}(k^2 + l^2 + 2kla^m)] \ . \end{aligned} \tag{2.7}$$

The odd moments are equal to zero.

In particular, for $k = 2$, $k = 4$ and $k = l = 2$ and $m = 1$, we get as in Jacquier, Polson and Rossi (1994):

$$\mu_2(\theta_1) = E(u_t^2) = r_y^2 \exp[r_w^2/2(1-a^2)] \ , \tag{2.8}$$

$$\mu_4(\theta_1) = E(u_t^4) = 3r_y^4 \exp[2r_w^2/(1-a^2)] \ , \tag{2.9}$$

and

$$\mu_{2,2}(1|\theta_1) = E[u_t^2 u_{t-1}^2] = r_y^4 \exp[r_w^2/(1-a)] \ . \tag{2.10}$$

Solving the above moment equations corresponding to $k = 2$, $k = 4$ and $m = 1$ yields the following proposition.

Proposition 2.5 ESTIMATING EQUATIONS.
Under the assumptions of Proposition **2.4**, *we have:*

$$a = \frac{[\log(\mu_{2,2}(1|\theta_1)) - \log(3) - 4\log(\mu_2) + \log(\mu_4)]}{\log(\frac{\mu_4}{3(\mu_2)^2})} - 1 \ , \tag{2.11}$$

$$r_y = \frac{3^{1/4}\mu_2}{\mu_4^{1/4}} \ , \tag{2.12}$$

$$r_w = \left(\log(\frac{\mu_4}{3(\mu_2)^2})(1-a^2)\right)^{1/2} \ . \tag{2.13}$$

Given the latter proposition, it is easy to compute a method-of-moments estimator for $\theta_1 = (a, r_y, r_w)'$ replacing the theoretical moments by sample counterparts based on the residuals $\hat{u}_t$. Let $\hat{\theta}_T$ denote the method-of-moments estimator of θ_1. Typically, $E(u_t^2)$, $E(u_t^4)$ and $E(u_t^2 u_{t-1}^2)$ are approximated by:

$$\hat{\mu}_2 = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_t^2 \quad \hat{\mu}_4 = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_t^4 \ , \quad \hat{\mu}_2(1) = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_t^2\hat{u}_{t-1}^2$$

respectively. $\hat{\theta}_T$ is consistent and asymptotically normally distributed. See Dufour and Valéry (2004) for an exhaustive presentation of its asymptotic properties.

3. The two-factor SV model

A simple single-factor SV model appears to be sufficient to capture the salient properties of volatility such as randomness and persistence. It is the shape of the conditional distribution of financial returns that is the problem [see Chernov *et al* (2003), Durham (2004a, 2004b)]. Standard SV models cannot match the high conditional kurtosis of returns (tail thickness) documented in the financial literature, of which equities are the most prominent example. Trying to capture nonlinearities in financial returns presents important implications for risk management and option pricing.

Hence, in this section we consider a two-factor specification driving the dynamic of the volatility process of the following form:

$$y_t - \mu_y = c(y_{t-1} - \mu_y) + \exp(w_t/2 + \eta_t/2) r_y z_t \ , \quad |c| < 1 \tag{3.14}$$

$$w_t = a_w w_{t-1} + r_w v_{1t} \ , \quad |a_w| < 1 \ , \tag{3.15}$$

$$\eta_t = b_\eta \eta_{t-1} + r_\eta v_{2t} \ , \quad |a_\eta| < 1 \ , \tag{3.16}$$

with $corr(v_{1t}, v_{2t}) = \rho_{12}$. The perturbations $v_{it}, \ \ i = 1, 2$ are $N(0, 1)$ and independent of z_t. We shall call the above model represented by equations (3.14)-(3.16) the autoregressive stochastic volatility model with two factors. Let $\theta_2 = (a_w, r_y, r_w, b_\eta, r_\eta, \rho_{12})'$ denote the parameter corresponding to the two-factor SV model. We derive the moment conditions used in a just-identified GMM framework which are stated in the proposition below.

Proposition 3.1 Moment conditions for the two-factor SV model.

$$\mu_2(\theta_2) = E(u_t^2) = r_y^2 \exp\left(\frac{1}{2}\frac{r_w^2}{1-a_w^2} + \frac{1}{2}\frac{r_\eta^2}{1-a_\eta^2} + \frac{r_w r_\eta \rho_{12}}{1-a_w a_\eta}\right), \tag{3.17}$$

$$\mu_4(\theta_2) = E(u_t^4) = 3r_y^4 \exp\left(2\frac{r_w^2}{1-a_w^2} + 2\frac{r_\eta^2}{1-a_\eta^2} + 4\frac{r_w r_\eta \rho_{12}}{1-a_w a_\eta}\right), \tag{3.18}$$

$$\mu_6(\theta_2) = E(u_t^6) = 15r_y^6 \exp\left(\frac{9}{2}\frac{r_w^2}{1-a_w^2} + \frac{9}{2}\frac{r_\eta^2}{1-a_\eta^2} + 9\frac{r_w r_\eta \rho_{12}}{1-a_w a_\eta}\right), \tag{3.19}$$

$$\mu_{2,2}(1|\theta_2) = E[u_t^2 u_{t-1}^2] = r_y^4 \exp(\frac{\sigma^2}{2}), \tag{3.20}$$

$$\mu_{4,4}(1|\theta_2) = E[u_t^4 u_{t-1}^4] = 9r_y^8 \exp(2\sigma^2), \tag{3.21}$$

$$\mu_{6,6}(1|\theta_2) = E[u_t^6 u_{t-1}^6] = 225r_y^{12} \exp(\frac{9}{2}\sigma^2), \tag{3.22}$$

where

$$\sigma^2 = 2\frac{r_w^2}{1-a_w^2} + 2\frac{r_\eta^2}{1-a_\eta^2} + 4\frac{r_w r_\eta \rho_{12}}{1-a_w a_\eta} + 2a_w\frac{r_w^2}{1-a_w^2} + 2\frac{a_w r_w r_\eta \rho_{12}}{1-a_w a_\eta} + 2r_w r_\eta \rho_{12} + 2\frac{a_\eta r_w r_\eta \rho_{12}}{1-a_w a_\eta} + 2a_\eta\frac{r_\eta^2}{1-a_\eta^2}. \tag{3.23}$$

These moment conditions will serve as a basis in the GMM set up for testing the number of SV factors in the volatility process. In a multivariate system, some authors did focus on testing for the number of SV factors in modelling the conditional heteroskedasticity in multivariate GARCH models. More specifically, Lanne and Saikkonen (2002) derive rank tests for the number of factors in an orthogonal GARCH system introduced by Alexander (2001) as a generalization of the GARCH factor model [see Engle (1984), Engle, Ng and Rothschild (1990), ...] to a multifactor model with orthogonal factors. More recently, Quintos (2005) extends Lanne and Saikkonen rank tests by allowing not only k factors that are conditionally heteroskedastic but also the remaining $p-k$ factors in a p-variate system but with less persistence.

4. Specification test for the one-factor SV model

In this section we propose a specification test to test the null hypothesis of linearity in the volatility process as stated in equation (2.4) against the alternative of a fractionally integrated Gaussian process for the volatility where equation (2.4) is replaced by:

$$(1-B)^d w_t = \eta_t \quad , \quad \eta_t \overset{i.i.d.}{\sim} N(0, \sigma_\eta^2) \tag{4.24}$$

where $d \in (-0.5, 0.5)$. When d is restricted to this domain, w_t is stationary and invertible [see Hosking (1981)]. By denoting $v_t(\tilde{\theta}) = \exp(w_t/2) r_y z_t$ where $\tilde{\theta} = (d, r_y, \sigma_\eta^2)'$, we review the first two moments of $v_t(\tilde{\theta})$ obtained from properties of the log-normal distribution as it is stated in Breidt, Crato and de Lima (1998):

$$\mu_2(\tilde{\theta}) = E(v_t(\tilde{\theta})^2) = r_y^2 \exp[\gamma(0)/2] \ , \tag{4.25}$$

$$\mu_4(\tilde{\theta}) = E(v_t(\tilde{\theta})^4) = 3 r_y^4 \exp[2\gamma(0)] \ , \tag{4.26}$$

and

$$\mu_{2,2}(h|\tilde{\theta}) = E[v_t(\tilde{\theta})^2 v_{t-h}(\tilde{\theta})^2] = r_y^4 \exp[\gamma(0)(1+\rho(h))] \ , \tag{4.27}$$

where the auto-covariance and autocorrelation functions for the long-memory process $\{w_t\}$, denoted by $\gamma(\cdot)$ and $\rho(\cdot)$ are given by:

$$\gamma(0) = \sigma_\eta^2 \frac{\Gamma(1-2d)}{\Gamma^2(1-d)} \ , \tag{4.28}$$

$$\rho(h) = \frac{\Gamma(h+d)\Gamma(1-d)}{\Gamma(h-d+1)\Gamma(d)} \ , \quad , h = 1, 2, \ldots, \tag{4.29}$$

[see Brockwell and Davis, (1991), p.522]. Then, the likelihood-ratio-type test statistic for comparative fit that is investigated here is given by:

$$\tilde{\xi}_T^C = T[M_T^*(\hat{\theta}_T|\mathcal{M}_0) - M_T^*(\tilde{\theta}_T|\mathcal{M}_1)] \tag{4.30}$$

where

$$M_T^*(\theta|\mathcal{M}_i) \equiv [\bar{g}_T(\hat{U}) - \mu(\theta|\mathcal{M}_i)]' \hat{\Omega}^{*-1} [\bar{g}_T(\hat{U}) - \mu(\theta|\mathcal{M}_i)], \quad i = 0, 1 \tag{4.31}$$

to test the null hypothesis that the true model, denoted by $\mathcal{M}_0$ is the linear volatility process against the alternative $\mathcal{M}_1$ which is the fractionally integrated gaussian volatility process.

5. Hypotheses tests and confidence sets

In this section we shall set the framework for testing general hypotheses as $H_0 : \ F \in \mathcal{H}_0$, where $\mathcal{H}_0$ is a subset of all possible distributions for the stochastic volatility model (2.3)-(2.4), that is,

$$\mathcal{H}_0 \equiv \{F(.) \ : F(\bar{y}) = F_0(\bar{y}|\psi(\theta)) \ and \ \psi(\theta) = 0\} \ , \tag{5.32}$$

where $\psi(\theta)$ is a $p \times 1$ continuously differentiable function of θ. H_0 is usually abbreviated as: $H_0 : \ \psi(\theta) = 0$. The derivative of the constraints $P(\theta) = \frac{\partial \psi}{\partial \theta'}$ has full row rank. Let $\hat{\theta}_T$ be the unrestricted estimator and $\hat{\theta}_T^c$ the constrained estimator obtained by minimizing the following criterion under H_0:

$$M_T^*(\theta) \equiv [\bar{g}_T(\hat{U}) - \mu(\theta)]' \hat{\Omega}^{*-1} [\bar{g}_T(\hat{U}) - \mu(\theta)] \ . \tag{5.33}$$

The Wald statistic is defined as

$$\xi_T^W = T\psi(\hat{\theta}_T)'[\hat{P}(\hat{J}'\hat{I}^{-1}\hat{J})^{-1}]^{-1}\psi(\hat{\theta}_T) \tag{5.34}$$

where $\hat{P} = P(\hat{\theta}_T)$, $\hat{I} = I(\hat{\theta}_T) = \Omega^*(\hat{\theta}_T)$, $\hat{J} = J(\hat{\theta}_T) = \frac{\partial \mu}{\partial \theta'}(\hat{\theta}_T)$.

The score statistic is defined from the gradient of the objective function with respect to θ evaluated at the constrained estimator. This gradient is given by:

$$\mathcal{D}_T = \frac{\partial \mu'}{\partial \theta}(\hat{\theta}_T^c)\hat{\Omega}^{*-1}(\mu(\hat{\theta}_T^c) - \bar{g}_T(\hat{U})) \tag{5.35}$$

and the test statistic is given by

$$\xi_T^S = T\mathcal{D}_T'(\hat{J}_0'\hat{I}_0^{-1}\hat{J}_0)^{-1}\mathcal{D}_T \ , \tag{5.36}$$

where $\hat{I}_0 = I(\hat{\theta}_T^c) = \Omega^*(\hat{\theta}_T^c)$, $\hat{J}_0 = J(\hat{\theta}_T^c) = \frac{\partial \mu}{\partial \theta'}(\hat{\theta}_T^c)$. Finally, we can introduce the difference between the optimal values of the objective function that we shall call the LR-type test in the simulations:

$$\xi_T^C = T[M_T^*(\hat{\theta}_T^c) - M_T^*(\hat{\theta}_T)] \ . \tag{5.37}$$

The three standard test statistics ξ_T^W, ξ_T^S, and ξ_T^C are known to be asymptotically equivalent and to follow a χ^2 distribution under the null hypothesis.

We also consider the $C(\alpha)$-type test statistic defined by:

$$PC(\tilde{\theta}_T^c) = T[\mu(\tilde{\theta}_T^c) - \bar{g}_T(\hat{U})]'\tilde{W}_0[\mu(\tilde{\theta}_T^c) - \bar{g}_T(\hat{U})] \tag{5.38}$$

where $\tilde{W}_0 = \tilde{I}_0^{-1}\tilde{J}_0(\tilde{J}_0'\tilde{I}_0^{-1}\tilde{J}_0)^{-1}\tilde{P}_0'[\tilde{P}_0(\tilde{J}_0'\tilde{I}_0^{-1}\tilde{J}_0)^{-1}\tilde{P}_0']^{-1}\tilde{P}_0(\tilde{J}_0'\tilde{I}_0^{-1}\tilde{J}_0)^{-1}\tilde{J}_0'\tilde{I}_0^{-1}$, with $\tilde{J}_0 = J(\tilde{\theta}_T^c) = \frac{\partial \mu}{\partial \theta'}(\tilde{\theta}_T^c)$, $\tilde{I}_0 = I(\tilde{\theta}_T^c) = \Omega^*(\tilde{\theta}_T^c)$, and $\tilde{P}_0 = P(\tilde{\theta}_T^c)$. $\tilde{\theta}_T^c$ is any root-n consistent estimator of θ that satisfies $\psi(\tilde{\theta}_T^c) = 0$. For our concern, $\tilde{\theta}_T^c$ will be obtained by imposing the constraints in the analytic expressions of the unrestricted method-of-moments estimator $\hat{\theta}_T$ given at equations (2.11) to (2.13), yielding a consistent restricted estimator without any optimization step. It is known [see Dufour and Trognon (2001, p.8, Proposition 3.1)] that the $C(\alpha)$-type test statistic is asymptotically distributed as a χ^2 variable under the null hypothesis.

In the simulations, we will focus on a particular form of the constraint, i.e. $\psi(\theta) = (1,0)\begin{pmatrix}\theta_{s1}\\ \theta_{s2}\end{pmatrix} = \theta_{s1}$ and the null hypothesis $H_0: \ \psi(\theta) = 0$ simplifies to $H_0: \ \theta_{s1} = 0$, (e.g. $\theta_{s1} \equiv a_w$, $\theta_{s1} \equiv (a_w, r_w)'$). We shall discuss at this stage a few anomalies arising when testing the joint null hypothesis of no heteroskedasticity $H_0: \ (a_w, r_w)' = \underline{0}$ against an alternative of stochastic volatility. We shall stress two interesting findings. The first one is when trying implementing the null hypothesis no heteroskedasticity, the score-type test statistics such as the score statistic and the $C(\alpha)$ statistic become identically null by construction through the derivatives of the moments of the volatility process. In that sense, the score-type test statistics are no longer meaningful under weaker regularity conditions.

As a consequence, the test of no heteroskedasticity against an alternative of stochastic volatility is performed by means of the Wald statistic and the LR-type statistic. However, some standard regularity conditions does not hold anymore when implementing the null hypothesis of homoskedasticity. Indeed, under the null, the moment conditions defining the estimator of the parameter of the one-factor SV model become nonlinearly redundant, that is the three moment conditions (2.8), (2.9) and (2.10) reduces to only one relevant moment condition. Furthermore, when testing $H_0 : \ (a_w, r_w)' = \underline{0}$ in the two-factor setup, which correspond to test one factor against two SV factors, the correlation parameter ρ_{12} become unidentified.

In such situations, it is known that the standard asymptotic theory does not provide reliable inference any longer. A simulation exercise performed later on in the paper strongly highlights the failure of the asymptotic theory when identification requirements are violated. In particular, the Wald statistic will exhibit severe size distortions for any length of the simulated path. As for the LR-type statistic, it will also demonstrate size distortions but in a lesser extent and remains globally valid in presence of identification problems. Indeed, it is known [see Dufour (1997)] that the Wald statistic is not reliable in models *locally almost unidentified* whereas the LR statistic still provides reliable inference. In such a context, simulation-based inference such as the technique of Monte Carlo tests presented in the next section, is one solution to correct for these extreme size distortions observed for its asymptotic counterparts. The other solution but not the easiest one consists in deriving the null and alternative asymptotic distributions under nonstandard regularity conditions as proposed by Andrews (2001).

We also provide confidence sets by inverting the test statistics. Let $S_0 = S(\psi, \bar{y})$ note one of the four previous tests statistics computed from the sample points $\bar{y} = (y_1, \ldots, y_T)$ and under the hypothesis $H_0 : \ \psi(\theta) = 0$. It is known that there is a correspondence between confidence sets and tests. The acceptance region of the hypothesis test, the set in the *sample space* for which $H_0 : \ \psi(\theta) = 0$ is accepted, is given by

$$A(\psi) = \{\bar{y} = (y_1, \ldots, y_T) \ : \ S(\psi, \bar{y}) \leq \chi^2_{1-\alpha}\} \tag{5.39}$$

for a α level test, and the confidence set, the set in the *parameter space* with plausible values of $\psi(\theta)$, is given by

$$C(y_1, \ldots, y_T) = \{\psi(\theta) \ : \ S(\psi, \bar{y}) \leq \chi^2_{1-\alpha}\} = \{\psi(\theta) \ : \ G(S(\psi, \bar{y})) \geq \alpha\} \ , \tag{5.40}$$

where $G(.)$ denotes the p-value function. These sets are connected to each other by the tautology

$$(y_1, \ldots, y_T) \in A(\psi) \Leftrightarrow \psi(\theta) \in C(y_1, \ldots, y_T) \ .$$

The hypothesis test fixes the parameter and asks what sample values (the acceptance region) are consistent with that fixed value. The confidence set fixes the sample value and asks what parameter values (the confidence set) make this sample value most plausible. Thus, if $A(\psi)$ is an acceptance region with level α, we have:

$$P_F[Y \notin A(\psi)] \leq \alpha \quad , \forall F \in \mathcal{H}_0$$

and hence,

$$P_F[Y \in A(\psi)] \geq 1 - \alpha \quad , \forall F \in \mathcal{H}_0 \ .$$

Then, the coverage probability of the set $C(Y)$ is given by:

$$P_F[\psi(\theta) \in C(Y)] = P_F[Y \in A(\psi)] \geq 1 - \alpha$$

showing that $C(Y)$ is a $1 - \alpha$ confidence set for $\psi(\theta)$.

Following this methodology, we will build confidence sets for the autoregressive parameter of the volatility process by retaining all the values of the parameter for which the p-value function is greater than or equal to $1 - \alpha$, yielding a $(1 - \alpha)$-level confidence set.

6. Monte Carlo testing

The technique of Monte Carlo tests has originally been suggested by Dwass (1957) for implementing permutation tests, and did not involve nuisance parameters. This technique has been later extended by Barnard (1963) and Birnbaum (1974). This technique has the great attraction of providing *exact* (randomized) tests based on any statistic whose finite sample distribution may be intractable but can be simulated.

We review in this section the methodology of Monte Carlo tests as it is exposed in Dufour (2002) where the distribution of the test statistic S may depend on nuisance parameters. For the test statistics exposed in section 5, their asymptotic distribution is asymptotically pivotal (Chi-square distribution), but their finite sample distribution remains unknown. At this stage, we need to make an effort of formalization to clearly expose the procedure. We consider a family of probability spaces $\{(\mathcal{Z}, \mathcal{A}_{\mathcal{Z}}, P_\rho) : \rho \in \Omega\}$ and suppose that S is a real valued $\mathcal{A}_{\mathcal{Z}}$-measurable function whose distribution is determined by $P_{\bar{\rho}}$ where $\bar{\rho}$ is the "true" parameter vector. We wish to test the hypothesis

$$H_0 : \bar{\rho} \in \Omega_0,$$

where Ω_0 is a nonempty subset of Ω. We consider a critical region of the form $S \geq c$, where c is a constant which does not depend on ρ. The critical region $S \geq c$ has *level* α if and only if

$$P_\rho[S \geq c] \leq \alpha, \forall \rho \in \Omega_0,$$

or equivalently,

$$\sup_{\rho \in \Omega_0} P_\rho[S \geq c] \leq \alpha.$$

Furthermore, $S \geq c$ has *size* α when

$$\sup_{\rho \in \Omega_0} P_\rho[S \geq c] = \alpha.$$

We consider a real random variable S_0 and random vectors of the form

$$S(N,\rho) = (S_1(\rho), \ldots, S_N(\rho))', \rho \in \Omega,$$

all defined on a common probability space $(\mathcal{Z}, \mathcal{A}_{\mathcal{Z}}, P)$, such that the variables $S_0, S_1(\bar{\rho}), \ldots, S_N(\bar{\rho})$ are i.i.d. or exchangeable for some $\bar{\rho} \in \Omega$, each one with distribution function $F[x|\bar{\rho}] = P[S_0 \leq x]$. Typically, S_0 will refer to a test statistic computed from the observed data when the true parameter vector is $\bar{\rho}$ (i.e., $\rho = \bar{\rho}$), while $S_1(\rho), \ldots, S_N(\rho)$ will refer to i.i.d replications of the test statistic obtained independently (e.g., by simulation) under the assumption that the parameter vector is ρ (i.e., $P[S_i(\rho) \leq x] = F[x|\rho]$). In other words, the observed statistic S_0 is simulated by first generating an "observation" vector y according to

$$y = g(\rho, z, v) \tag{6.41}$$

where the function g has the bivariate AR(1)-SV specification as stated in equations (2.3) and (2.4), with $\rho = (c, \mu_y, \theta)'$, $\theta = (a, r_y, r_w)'$. The perturbations z and v have known distributions, which can be simulated ($N(0,1)$ or student, or mixtures, e.g.). We can then compute

$$S(\rho) \equiv S[g(\rho, z, v)] \equiv g_S(\rho, z, v)\ . \tag{6.42}$$

The observed statistic S_0 is then computed as $S_0 = S[g(\bar{\rho}, z_0, v_0)]$ and the simulated statistics as $S_i(\rho) = S[g(\rho, z_i, v_i)]$, $i = 1, \ldots, N$ where the random vectors $z_0, z_1, \ldots, z_N$ are i.i.d. (or exchangeable) and $v_0, v_1, \ldots, v_N$ are i.i.d. (or exchangeable) as well.

The technique of Monte Carlo tests provides a simple method allowing one to replace the theoretical distribution $F(x|\rho)$ by its sample analogue based on $S_1(\rho), \ldots, S_N(\rho)$:

$$\hat{F}_N[x; S(N,\rho)] = \frac{1}{N}\sum_{i=1}^{N} s(x - S_i(\rho)) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{[0,\infty]}(x - S_i(\rho))$$

where $s(x) = 1_{[0,\infty]}(x)$ and $1_A(x)$ is the indicator function associated with the set A. We also consider the corresponding sample tail area function:

$$\hat{G}_N[x; S(N,\rho)] = \frac{1}{N}\sum_{i=1}^{N} s(S_i(\rho) - x)\cdot$$

and the p-value function

$$\hat{p}_N[x|\rho] = \frac{N\hat{G}_N[x|\rho] + 1}{N+1}\ .$$

The sample distribution function is related to the ranks $R_1, \cdots, R_N$ of the variables

$S_1(\rho), \ldots, S_N(\rho)$ (when put in ascending order) by the expression:

$$R_j = N\hat{F}_N[S_j; S(N,\rho)] = \sum_{i=1}^{N} s(S_j(\rho) - S_i(\rho)),\ j = 1, \ldots, N.$$

The central property which is exploited here is the following: to obtain critical values or compute p-values, the "theoretical" null distribution $F[x|\bar{\rho}]$ can be replaced by its simulation-based "estimate" $\hat{F}_N[x|\rho] \equiv \hat{F}_N[x; S(N,\rho)]$ in a way that will preserve the level of the test in *finite samples, irrespective of the number N of replications used.* At this stage we shall refer the reader to Dufour (2002, p.13, Proposition 4.1) in which the author states the finite sample validity of Monte Carlo tests when the p-value function is maximized over the entire set of the nuisance parameters.

Therein, the author shows that the critical region $\sup\{\hat{G}_N[S_0|\rho] : \rho \in \Omega_0\} \leq \alpha_1$ has level α irrespective of the presence of nuisance parameters in the distribution of the test statistic S under the null hypothesis $H_0 : \bar{\rho} \in \Omega_0$. Likewise, the (almost) equivalent randomized critical regions $\inf\{\hat{F}_N[S_0|\rho] : \rho \in \Omega_0\} \geq 1 - \alpha_1$ or $S_0 \geq \sup\{\hat{F}_N^{-1}[1 - \alpha_1|\rho] : \rho \in \Omega_0\}$ are shown to have the same level α as their non-randomized analogues. Dufour (2002) calls such tests maximized Monte Carlo (MMC) tests. The function $\hat{G}_N[S_0|\rho]$ (or $\hat{p}_N[S_0|\rho]$) is then maximized with respect to $\rho \in \Omega_0$, keeping the observed statistic S_0 and the simulated disturbance vectors $z_1, ..., z_N$ and $v_1, ..., v_N$ fixed. The function $\hat{G}_N[S_0|\rho]$ is a step-type function which typically has zero derivatives almost everywhere, except on isolated points (or manifolds) where it is not differentiable. So it cannot be maximized with usual derivative-based algorithms. However, the required maximizations can be performed by using appropriate optimization algorithms that do not require differentiability, such as *simulated annealing.* For further discussion of such algorithms, the reader may consult Goffe, Ferrier, and Rogers(1994).

On the other hand, Dufour (2002) also proposes simplified (asymptotically justified) approximate versions of Monte Carlo tests where the p-value function may be evaluated either at a consistent point estimate and defines thereby a Bootstrap version, or at a consistent set estimate of ρ and defines instead confidence-set-Monte Carlo tests. The author shows [see Dufour, (2002, p.16, Proposition 5.1 and p.19, Proposition 6.3)] that both tests are asymptotically valid in the sense that they have the correct level α asymptotically and the estimated p-values converge to the true p-values. He also assesses the validity of the MMC tests and the asymptotic Monte Carlo tests based on consistent set estimators for general distributions , when ties have non-zero probability [see Dufour, (2002, p.14, Proposition 4.2 and p.17, Proposition 5.2)].

In the remaining of the paper we will implement the maximized and Bootstrap versions of the Monte Carlo technique and investigate in a comparative Monte Carlo study their actual size and power performances with respect to those of the standard asymptotic tests developed in section 5.

7. Simulation results

In this section we assess by simulations the actual sizes of the test procedures described earlier in the paper. The nominal level of the tests has been set to $\alpha = 5\%$. M represents the number of replications to assess the actual size of the test, and N represents the number of simulated statistics used in the Monte Carlo tests. T is the sample size of the series y_t whose data generating process is assumed to be specified as in equations (2.3)-(2.4) for the one-factor SV model and as in equations (3.14)-(3.16) for the two-factor SV model. Implementation is performed with the GAUSS software version 3.2.37. Note that the autoregressive parameters a_w and a_η in the autoregressive specifications of the volatility process are restricted to (-1,1) to ensure stationarity of the volatility process. At this end, each time the estimate of a falls outside of its domain we truncate the estimator by setting it to $a = 0.99$ when $a >= 1$ and to $a = -0.99$ when $a <= -1$.

The Wald statistic as defined at equation (5.34) is evaluated at the unrestricted method-of-moments estimator $\hat{\theta}_{1T}$. The Score statistic as defined at equation (5.36) is evaluated at the restricted estimator $\hat{\theta}^c_T$ which minimizes the criterion $M^*_T(\theta)$ defined at equation (5.33) submitted to the constraint $a_w = 0$ whereas $\tilde{\theta}^c_T$ represents another restricted estimator of θ obtained by setting $a_w = 0$ in the analytic expressions of the unrestricted method-of-moments estimator $\hat{\theta}_T$ given at equations (2.11)- (2.13). The $C(\alpha)$-type statistic as defined at equation (5.38) is evaluated at this restricted estimator $\tilde{\theta}^c_T$ of θ. Further, the LR-type test statistic corresponds to the difference between the optimal values of the objective function. Let $LR(\hat{\Omega}) \equiv \xi^C_T$ [see equation (5.37)] where $\hat{\Omega} \equiv \Omega(\hat{\theta}_T)$. The weighting matrix $\hat{\Omega}$ is estimated by a kernel estimator with a fixed-Bandwith Bartlett Kernel, where the lag truncation parameter K has been set to $K = 2$.

Let S denote the test statistic which alternately takes the form of one of the four test statistics earlier mentioned and let S_0 denote the statistic computed from the "pseudo-true" data obtained by simulation under the true data generating process. The critical regions used to perform the tests are of the form:

$$\mathcal{R}_c = \{S_0 > \chi^2_{1-\alpha}(i)\}, \quad i = 1, 2, 3$$

for the standard asymptotic tests, and of the form:

$$\mathcal{R}_c = \{\hat{p}_N[S_0|\hat{\rho}^c_T] \leq \alpha\}$$

with the p-value function

$$\hat{p}_N[S_0|\rho] = \frac{N\hat{G}_N[S_0|\rho] + 1}{N + 1} \ ,$$

and the tail area function

$$\hat{G}_N[S_0; S(N,\rho)] = \frac{1}{N}\sum_{i=1}^{N} s(S_i(\rho) - S_0),$$

for the Monte Carlo tests where the p-value function may be evaluated at any consistent restricted estimator of $\rho = (c, \mu_y, \theta')'$ where μ_y has been set to zero everywhere. $\theta = \theta_1 = (c, a_w, r_y, r_w)'$ refers to the parameter of the one-factor SV model while $\theta = \theta_2 = (c, a_w, r_y, r_w, a_\eta r_\eta, \rho_{12})'$ refers to the parameter of the two-factor SV model. The simulated statistics $S_i(\rho)$ $i = 1, ..., N$ will always be evaluated under the null hypothesis in the Monte Carlo tests whatever the hypothesis to be tested. The nominal level α has been set to $\alpha = 5\%$. Monte Carlo (MC) tests whose p-value function is evaluated at a consistent point estimate of the nuisance parameters follow the methodology presented in section 6. In practice, to implement maximized MC tests (MMC), we maximize the p-value over a grid with increment of 0.1 around a consistent estimate of the nuisance parameter under the null.

7.1. Size investigation

We study the actual size of the various tests and compare them to their nominal size fixed at $\alpha = 5\%$. Concerning the specification test, we study in Table 1 the actual size of rejecting the null hypothesis of a linear autoregressive volatility specification against an alternative fractionally integrated gaussian volatility process. The parameters have been set to $c = 0.3$, $r_y = r_w = 0.5$ and the autoregressive volatility parameter $a_w = 0.3$. As usually encountered in specification tests, the test underrejects the null in small samples and requires at least $T = 5000$ observations to reach the nominal level stated at $\alpha = 5\%$.

The results reported in the top part of Table 2 for rejecting the null hypothesis $H_0 : a_w = 0$ display evidence for the asymptotic tests of under-rejecting H_0 for the Wald and the $C(\alpha)$ tests particularly in small samples, whereas the score-type and the LR-type tests tend to over-reject. In particular the underrejection under the null tends to induce a loss of power under the alternative. By contrast, we can see in the bottom part of Table 2 that the technique of MC tests achieves in correcting for the size distortions of the asymptotic tests. We also investigate in Table 3, a joint test of homoskedasticity in the stochastic volatility model by testing the null hypothesis $H_0 : a_w = 0, r_w = 0$ by means of the Wald-type and LR-type statistics. The score-type test statistics have been evacuated here since they are identically null by construction. The asymptotic critical value is given by the 95%-quantile of the chi-square distribution with two degrees of freedom which correspond to $c_2 = 5.99$. Note the extremely huge over-rejection (more than 90%) displayed by the asymptotic Wald test when usual regularity conditions are not satisfied. Whatever sample size is considered, the situation is not getting better. Concerning the LR statistic behavior, it tends to slightly overreject in small samples and underreject in large samples. Once again we can note in Table 3, that Monte Carlo tests achieve in correcting the severe size distortions observed for the asymptotic tests. More specifically, the Wald statistic performs extremely poorly for the joint null hypothesis $H_0 : a_w = 0, r_w = 0$ whereas the LR statistic is more reliable. Indeed, the estimators used to construct the test statistics, are based on the moments of the volatility process but under this joint null hypothesis these moment conditions become nonlinearly redundant. As a consequence, the Jacobian of the moment conditions is no more of full-column rank under the null causing some singularity issue for the covariance matrices. It is known [see Dufour (1997)] that the Wald statistic is not reliable under nonregular conditions whereas the LR statistic still provides reliable inference. It is worth noting in Table 4 that when Monte Carlo tests (MC) evaluated at a consistent restricted estimate of the nuisance parameter fail to correct for the size distortions observed in small samples ($T = 50, 100$) for the LR statistic, its maximized version (MMC) does correct for the size distortions. Indeed, we observe in Table 4 that MMC test achieves in reaching the correct level stated at $\alpha = 5\%$ in small samples ($T = 50, 100$) whereas MC tests remains around 10%. The MMC version is performed by maximizing the p-value function on a neighborhood of the restricted estimate of the nuisance parameters which are c and r_w. Symmetrically, we observe that MC tests correct for underrejection displayed by the LR statistic in large samples. On the other hand, the results shown in Table 4 give evidence on the poor performance of the Wald statistic under nonstandard regularity conditions and provide a striking illustration of the theoretical results obtained in Dufour (1997).

Finally, we also study in Table 5 the size of the test of one factor against two SV factors. The severe overrejections are striking and unfortunately are getting worse and worse as the sample size increases. At $T = 2000$, the asymptotic test based on the standard χ^2 distribution rejects $H_0 : \quad a_w = r_w = 0$ 98.8% of the times for the Wald statistic while it reaches 69.2% for the LR statistic. More interestingly, we observe that this time the MC tests cannot correct for size distortions. Indeed. Bootstrap tests which are known to rely on strong regularity conditions collapse in presence of identification problems. More specifically, under $H_0 : \quad a_w = r_w = 0$, the correlation parameter ρ_{12} between the perturbation s of the two SV factors become unidentified. This parameter has been set to a low level of correlation, namely $\rho_{12} = 0.3$ to justify the usefullness of a second factor. In situations of (almost) nonidentification, standard asymptotic distributions become invalid. Andrews (2001) derives asymptotic results under nonstandard conditions, in particular when the GARCH AR parameter becomes unidentified under a null of homoskedasticity.

7.2. Power investigation

Here we study the actual power of the different tests. Note that the standard asymptotic tests for testing the null hypothesis $H_0 : a_w = 0$ have been corrected for size distortions using the corresponding simulated critical values computed on $M = 10,000$ replications, as reported in Table 7 which yields exact 5%-level tests under the null hypothesis. Concerning the specification test, to simulate the model under the alternative of a fractionally integrated gaussian process, we follow Bollerslev and Mikkelsen (1996) and truncate the moving average filter and then let the process run for a long while to attenuate the effects of transients. Bollerslev and Mikkelsen suggest to truncate at $k = 1000$ but since the moving average coefficients become very small after 160, we chose to truncate at $k = 160$ yielding the moving average filter $\sum_{k=0}^{160} \psi_k B^k$. We then trim off the first 10000 observations. All parameters have been kept to the same values as under the null hypothesis with the long memory parameter $d = 0.3$ replacing the autoregressive parameter $a_w = 0.3$. We then observe in Table 6 that the simulations averaged over 1000 replications, require at least 1000 observations to exhibit sufficient power. Note also that the Monte Carlo tests do gradually loose power when compared to their asymptotic analogues due to some noise introduced by lengthy simulations.

In Table 8, we study the capability of the test of rejecting the null of $H_0 : a_w = 0$. We observe that both inference techniques, that is the asymptotic and Monte Carlo tests, suffer from a lack of power when the sample sizes are very small ($T = 50, 100, 200$). Note also the increase in power when we switch from one type of alternative: $H_1 : a_w = 0.8$ to a more persistent one: $H_1 : a_w = 0.99$. The power of Monte Carlo tests could be improved in small samples by increasing the number of simulated statistics from $N = 99$ to $N = 299, 499, 999$. Note that although the asymptotic procedure seems in some cases to exhibit more power w.r.t. Monte Carlo tests, the former however remains a not feasible benchmark for real data whose data generating process (DGP) is generally unknown. In this respect the simulation-based inference technique appears more robust to any DGP. Both test procedures have more power when the sample size grows which is intuitive since both tests are asymptotically justified. Further, note that the $C(\alpha)$ test demonstrates more power than its competitors at any sample sizes. In particular the $C(\alpha)$ test performs better than the score test statistic whereas both belong to the same score-type family. The $C(\alpha)$ test statistic has besides the advantage of being the easiest to implement since it does not require in our case any optimization procedure. Indeed the restricted estimate of θ is obtained by simply imposing the constraint in the analytical expressions available for the unrestricted moment estimator.

We also examine in Tables 9 and 10, the power of the joint test of the null hypothesis of homoskedasticity against the alternative $H_1 : a = 0.5, r_w = 0.5$ in the one SV factor framework. The Wald-type test has little power compared to the LR-type test which still remains valid under nonstandard conditions. Indeed, the Wald test after being corrected for the size distortions, is not consistent at all when increasing the sample size. In this respect, it is known [see Dufour (1997)], that Wald tests are not reformable in nonstandard situations, whatever asymptotic, Monte Carlo or *maximized* MC tests, exhibit the same inconsistent behavior for the Wald test. By contrast, the LR-type test remains consistent despite some singularity issues, even though its finite and asymptotic distribution may be modified.

We further study the power of the test of one factor against two SV factors in Table 11. We observe that the test has little power. The increase in power with the sample size is very slow and does not exceed 39% at $T = 2000$ for the LR statistic. Once again, we observe that the Wald statistic leads to an inconsistent test under identification problems since the rejection of the null hypothesis decreases with the sample size. This illustrates once more the invalid feature of conducting inference with Wald statistics under nonregular conditions. Unlike the Wald statistic, the LR statistic remains valid in such situations even though its asymptotic distribution may be modified. Furthermore, the LR statistic is preferred over the Wald statistic for its robustness (invariance) properties [see Dagenais and Dufour (1991)].

Finally, we also provide some plots of the power functions for asymptotic (in dashed line) and Monte Carlo (in cubic line) Wald and LR tests in Figure 1, and for score-type and $C(\alpha)$-type tests in Figure 2, respectively. Once again, we observe that the $C(\alpha)$ test has more power than its counterparts and displays a much smoother power function when compared to the tests involving the unrestricted estimator (the LR or the Wald tests). The score-type test also performs better than the LR or the Wald tests.

8. Empirical application

In this subsection we test the null hypothesis of no-persistence in the volatility, the hypothesis of linear specification for the volatility process against the alternative of a fractionally integrated specification, and also the null hypothesis of one SV factor against two SV factors, from real data (Standard and Poor's Composite Price Index (SP), 1928-87).

8.1. Data

The data have been provided by Tauchen where Efficient Method of Moments have been used by Gallant, Hsieh and Tauchen to fit a standard stochastic volatility model. The data to which we fit the univariate stochastic volatility model is a long time series comprised of 16,127 daily observations, $\{\tilde{y}_t\}_{t=1}^{16{,}127}$, on adjusted movements of the Standard and poor's Composite Price Index, 1928-87. The raw series is the Standard and Poor's Composite Price Index (SP),daily, 1928-87. We use a long time series, because, among other things, we want to investigate the long-term properties of stock market volatility through a persistence test. The raw series is converted to a price movements series, $100[\log(SP_t) - \log(SP_{t-1})]$, and then adjusted for systematic calendar effects, that is, systematic shifts in location and scale due to different trading patterns across days of the week, holidays, and year-end tax trading. This yields a variable we shall denote y_t.

8.2. Results

The unrestricted estimated value of ρ from the data is:

$$\hat{\rho}_T = (0.129,\ 0.926,\ 0.829,\ 0.427)'\ ,$$

$$\hat{\sigma}_T = [0.007,\ 2.89,\ 1.91,\ 8.13]'\ ,$$

where the method-of-moments estimated value of a corresponds to $\hat{a}_T = 0.926$. We may conjecture that there is some persistence in the data during the period 1928-87 what is statistically checked by performing the tests below. The restricted estimated values of ρ from the data are:

$$\hat{\rho}_T^c = (0.129,\ 0,\ 0.785,\ 1.152)'\ ,$$

$$\hat{\sigma}_T = [0.007,\ -, 1.95,\ 1.77]'\ ,$$

and

$$\tilde{\rho}_T^c = (0.129,\ 0, 0.829,\ 1.133)'\ ,$$

$$\hat{\sigma}_T = [0.007,\ -,\ 1.91,\ 1.66]'\ .$$

Note the large discrepancy between the unrestricted and restricted estimates of r_w where the restricted estimates are not consistent if the null hypothesis $H_0:\ a = 0$ is false.

In Table 12, we observe that all standard asymptotic tests reject indeed the null hypothesis of no-persistence in the volatility since $S_0 > \chi^2_{1-\alpha}(1) = 3.84$ as well as all the Bootstrap tests whose p-value is equal or less than 5%, whatever length of the simulated statistics is used to implement them. At this stage, it is well-known in the financial literature that financial returns display long-run dependence in the volatility, a feature which is not fully capture by standard SV models. With this respect, Bollerslev and Mikkelsen (1996), Ding, Granger and Engle (1993), Breidt, Crato and Lima (1998) present evidence that long-memory models like those of Granger and Joyeux (1980) might be needed to account for the high degree of persistence in financial volatility. That is the reason why we found it possibly informative to implement a specification test for a linear volatility against a fractionally differencing volatility process. Concerning this specification test, the results shown in the bottom part of Table 12 give evidence in favor of the null hypothesis of linear volatility against the alternative of a fractionally integrated volatility process as given by the statistic $\tilde{\xi}^C_T$ defined in equations (4.30) and (4.31). Indeed, the observed statistic ($\tilde{\xi}^C_T = 0.00345$) is much below the asymptotic critical value of $\chi^2_{95}(3) = 7.81$. The same hold for the MC p-values which are around 0.8 and greater than $\alpha = 0.05$. Such an observation might indicate that introducing a long-memory component in the volatility process is not well-suited or at least sufficient to capture the dynamic of financial returns. This evidence seems to give support for the presence of nonlinearities, to dependence in the tails which could be potentially captured by a second SV factor. To check for that, we implement the test for one SV factor against two SV factors by testing $H_0 : \quad a_w = r_w = 0$ in the two-factor SV framework. The first factor may act as a long-memory component while the second factor is expected to model the tail behavior. In Table 12, the asymptotic and bootstrap tests based on the LR statistic do reject the null of one factor in favor of two SV factors in the data, at all levels. By contrast, the Wald statistic once again yields controversial results. Indeed, the asymptotic test does reject the null whereas the bootstrap tests cannopt reject the null of one SV factor. This observation confirms the idea that inference based on Wald statistics does not produce reliable inference in nonstandard situations [see Dufour(1997)].

We also provide in Table 13 confidence sets by inverting the corresponding test statistics as exposed in section 5. The coverage probabilities for the confidence sets are $1 - \alpha = 95\%$. We can observe that all tests do cover the estimated value of a ,($\hat{a} = 0.926$), at the confidence level of 95%, except for the Bootstrap version of the score test statistic that covers at a lower confidence level of 93%. We may conclude by saying that the data seem to exhibit some persistence features as usually expected from financial data. But more interestingly, our empirical results seem to support the result obtained by Chernov *et alii* (2003) that two-factor SV models better accommodate the tail behavior of (conditional) return distributions and possibly capture some rapid moves in the dynamic of volatility during extreme market conditions.

9. Concluding remarks

The $C(\alpha)$ test outperforms the other types of tests while being the easiest to implement since it does not require in our framework any optimization procedure. It has good statistical properties: a good level and a high power for sufficiently large sample sizes. On the other hand, Monte Carlo tests and *maximized* MC tests appear as a good alternative to the standard asymptotic tests, specifically when the standard asymptotic approach fails - in situations of almost-unidentified models where the modified distribution of the test statistic remains unknown. We may consider as further research an extension of our approach to asymmetric and fat-tailed distributions such as the asymmetric student distribution and shall test the hypothesis of leverage effect in the stochastic volatility model.

Table 1. Size of asymptotic and Monte Carlo tests, specification test

LEVELS in % specification test								
	T=50		*T=100*		*T=200*		*T=500*	
	Asy	**MC**	**Asy**	**MC**	**Asy**	**MC**	**Asy**	**MC**
$LR(\hat{\Omega})$	0.2	0.3	0	0.1	0.1	0.1	0	0.1
	T=1000		*T=2000*		*T=5000*			
	Asy	**MC**	**Asy**	**MC**	**Asy**	**MC**	**Asy**	**MC**
$LR(\hat{\Omega})$	0	0.7	0.1	0.7	5.1	1.3	-	-

Table 2. Size of asymptotic and Monte Carlo tests, $,H_0: \ a = 0$

LEVELS in % (under $H_0: \ a_w = 0$)						
	Asymptotic tests					
	T=50	T=100	T=200	T=500	T=1000	T=2000
Wald	0.1	0.7	0.9	2.1	2.4	3.2
$Score(\hat{\Omega}_C)$	7.7	6	2.6	2.8	3.2	3
$LR(\hat{\Omega})$	7.5	4.8	3.8	2.5	3	3.7
$C(\alpha)$	0.4	0.7	2.6	3	2.9	2.9
	Monte Carlo tests					
	T=50	T=100	T=200	T=500	T=1000	T=2000
Wald	5.4	5.1	3	2.6	5.1	5.5
$Score(\hat{\Omega}_C)$	5.2	5.1	6	6	4.7	3
$LR(\hat{\Omega})$	4.2	5.6	5.8	6.6	5.5	4.8
$C(\alpha)$	4.7	4.4	6	6.9	5.4	4

Table 3. Size of asymptotic and Monte Carlo tests, $H_0: \ a_w = 0, r_w = 0$

LEVELS in % ($H_0 : a_w = 0, r_w = 0$),(nuisance:$c = 0.3, r_y = 0.5$)						
	Asymptotic joint tests					
	T=50	T=100	T=500	T=1000	T=2000	T=5000
Wald	94.8	91.6	90.7	90	90.2	92.3
$LR(\hat{\Omega})$	8.8	8.9	1.4	0.7	0.5	0.6
	Monte Carlo joint tests					
	T=50	T=100	T=500	T=1000	T=2000	T=5000
Wald	5.5	4.6	3.6	5.8	4.4	4.3
$LR(\hat{\Omega})$	8.1	7.3	4.7	4.5	3.2	4

Table 4. Size of asymptotic and Monte Carlo tests, $H_0: \ a_w = 0, r_w = 0$

LEVELS in % ($H_0 : a_w = 0, r_w = 0$), (nuisance:$c = 0.95, r_y = 0.5$)									
	T=50			*T=100*			*T=500*		
	Asy	**MC**	**MMC**	**Asy**	**MC**	**MMC**	**Asy**	**MC**	**MMC**
Wald	93.8	4.3	4.5	92.2	5	4.2	91.1	3	2.9
$LR(\hat{\Omega})$	9.4	10.5	3.3	8.2	9.9	5.2	1.50	6.4	4.9
	T=1000			*T=2000*			*T=5000*		
	Asy	**MC**	**MMC**	**Asy**	**MC**	**MMC**	**Asy**	**MC**	**MMC**
Wald	88.8	5.6	5	90.8	4.4	4.3	91	3.9	3.9
$LR(\hat{\Omega})$	0.6	5.6	4.1	0.4	3.2	3.1	0.6	4.7	4.1

Table 5. Size of asymptotic and Monte Carlo tests, $H_0 : a_w = 0, r_w = 0$, two-factor SV model

LEVEL in % ($H_0 : a_w = 0, r_w = 0$) (nuisance: $c = 0.95, r_y = 0.5, a_\eta = 0.7, r_\eta = 0.5, \rho_{12} = 0.3$)									
	T=50			*T=100*			*T=500*		
	Asy	**MC**	**MMC**	**Asy**	**MC**	**MMC**	**Asy**	**MC**	**MMC**
Wald	4.8	15.8	5	25.4	15.4	2	72.8	19.2	5.2
$LR(\hat{\Omega})$	19.6	7.4	2	35.2	10	1.6	60.8	5.6	2.4
	T=1000			*T=2000*			*T=5000*		
	Asy	**MC**	**MMC**	**Asy**	**MC**	**MMC**	**Asy**	**MC**	**MMC**
Wald	91.4	21.2	3.8	98.8	15.4	3	-	-	-
$LR(\hat{\Omega})$	65.2	6.2	2.8	69.2	15.8	2.1	-	-	-

Table 6. Power of asymptotic and Monte Carlo tests, specification test

POWER in % specification test								
	T=50		*T=100*		*T=200*		*T=500*	
	Asy	**MC**	**Asy**	**MC**	**Asy**	**MC**	**Asy**	**MC**
$LR(\hat{\Omega})$	7.2	1.5	2	1.8	0.4	8.4	6.8	26
	T=1000		*T=2000*		*T=5000*			
	Asy	**MC**	**Asy**	**MC**	**Asy**	**MC**	**Asy**	**MC**
$LR(\hat{\Omega})$	32.5	33.2	74.4	41.1	83.3	46.5	-	-

Table 7. Simulated critical values, under $H_0 :\ a_w = 0$

Simulated critical values						
	M=10,000 replications					
	T=50	T=100	T=200	T=500	T=1000	T=2000
Wald	0.8458	1.4295	2.8303	2.5826	2.7878	3.0203
$Score(\hat{\Omega}_C)$	1.7051	2.3336	2.6773	2.9260	2.9472	2.9523
$LR(\hat{\Omega})$	5.7228	3.7033	2.7759	3.0385	3.1352	2.9970
$C(\alpha)$	1.7974	2.3030	2.6901	2.8807	2.8879	2.9133

Table 8. Power of size-corrected asymptotic and Monte Carlo tests

POWER in % (under H_1)						
	Size-corrected Asymptotic tests $H_1 : a_w = 0.8$					
	T=50	T=100	T=200	T=500	T=1000	T=2000
Wald	10.9	17	23.4	60.4	84.5	93.2
$Score(\hat{\Omega}_C)$	16.8	25	47	78.6	93.9	97.8
$LR(\hat{\Omega})$	10.3	16.8	37.6	71.5	88.9	96.6
$C(\alpha)$	19.7	30.9	51.8	81.8	96	99.5
	$H_1 : a_w = 0.99$					
Wald	31.2	59.5	81.5	90.9	99	99.6
$Score(\hat{\Omega}_C)$	39.7	55.7	85.4	97.7	99.3	99.9
$LR(\hat{\Omega})$	25	44.6	77.3	96.7	99.2	99.3
$C(\alpha)$	41.5	68.8	91.6	99.2	99.7	100
	Monte Carlo tests ($N = 99$) $H_1 : a_w = 0.8$					
	T=50	T=100	T=200	T=500	T=1000	T=2000
Wald	10.1	11.8	19.4	44.8	68.3	84
$Score(\hat{\Omega}_C)$	15	18.2	27.9	63.3	89.7	96.8
$LR(\hat{\Omega})$	9.4	10	23.4	60.5	83.5	92.4
$C(\alpha)$	21.6	28.8	43.4	74.1	93.5	98.5
	$H_1 : a_w = 0.99$					
Wald	28.7	54.1	74.6	87.5	96.3	96.5
$Score(\hat{\Omega}_C)$	11.9	22.3	39.6	82.7	94.4	97.8
$LR(\hat{\Omega})$	15.8	29.8	55.6	72.6	98.5	99.2
$C(\alpha)$	36.1	62.6	78.8	91.6	99.6	99.9

Table 9. Power of asymptotic and Monte Carlo tests, $H_1 : a_w = 0.5, r_w = 0.5$, set I

POWER in % (under H_1)						
	Asymptotic joint tests $H_1 : a_w = 0.5, r_w = 0.5$					
	T=50	T=100	T=500	T=1000	T=2000	T=5000
Wald	15.8	17.6	18.1	12.7	6.7	1.3
$LR(\hat{\Omega})$	10.9	13.3	84.8	99.4	99.9	100
	Monte Carlo joint tests ($N = 499$)					
	T=50	T=100	T=500	T=1000	T=2000	T=5000
Wald	16.1	18.8	18	12.6	6.9	1.6
$LR(\hat{\Omega})$	14.5	15.7	86.5	99.1	99.9	100

Table 10. Power of asymptotic and Monte Carlo tests, $H_1 : a_w = 0.5, r_w = 0.5$, set II

POWER in % ($H_1 : a_w = 0.5, r_w = 0.5$), **(nuisance:** $c = 0.95, r_y = 0.5$)									
	T=50			*T=100*			*T=500*		
	Asy	**MC**	**MMC**	**Asy**	**MC**	**MMC**	**Asy**	**MC**	**MMC**
Wald	18	16.8	12.8	20.2	17.2	16.6	17.6	16.4	16.2
$LR(\hat{\Omega})$	11	14	3.8	15.4	17.4	11.6	84.6	85.6	85.4
	T=1000			*T=2000*			*T=5000*		
	Asy	**MC**	**MMC**	**Asy**	**MC**	**MMC**	**Asy**	**MC**	**MMC**
Wald	12	11.6	11.5	6	6	6	1	0.8	0.8
$LR(\hat{\Omega})$	99.6	99	99	100	100	100	100	100	100

Table 11. Power of asymptotic and Monte Carlo tests, $H_1 : a_w = 0.5, r_w = 0.5$, two-factor SV model

POWER in % ($H_1 : a_w = 0.5, r_w = 0.5$) (**nuisance:** $c = 0.95, r_y = 0.5, a_\eta = 0.7, r_\eta = 0.5, \rho_{12} = 0.3$)									
	T=50			*T=100*			*T=500*		
	Asy	**MC**	**MMC**	**Asy**	**MC**	**MMC**	**Asy**	**MC**	**MMC**
Wald	6.6	22	18.2	4.6	9.4	5.6	2	6	4.2
$LR(\hat{\Omega})$	7.6	9.8	5.8	9.4	12.8	5.8	18.6	20	11.8
	T=1000			*T=2000*			*T=5000*		
	Asy	**MC**	**MMC**	**Asy**	**MC**	**MMC**	**Asy**	**MC**	**MMC**
Wald	0.8	6	3.4	0.6	7	2.8	-	-	-
$LR(\hat{\Omega})$	28.4	30	18.2	31.6	38.8	25.2	-	-	-

Table 12. Empirical application

data				
$H_0 :\ a_w = 0$				
	Asymptotic tests	*Monte Carlo tests*		
	S_0	N=19	N=99	N=999
Wald	206.03	0.05	0.01	0.001
$Score(\hat{\Omega}_C)$	1039.04	0.05	0.01	0.001
$LR(\hat{\Omega})$	63.20	0.05	0.01	0.001
$C(\alpha)$	854.55	0.05	0.01	0.001
specification test				
$\tilde{\xi}_T^C$	0.00345	0.80	0.80	0.789
test of one against two SV factors				
Wald	522.55	0.25	0.21	0.18
$LR(\hat{\Omega})$	3.95e8	0.05	0.01	0.001

Table 13. Confidence sets

Confidence sets for a, $(1-\alpha = 95\%)$		
	Asymptotic	*Monte Carlo*
Wald	]0.92,0.93]	[0.92,0.93]
$Score(\hat{\Omega}_C)$	]0.92,0.93]	[0.92,0.93]*
$LR(\hat{\Omega})$	]0.92,0.93]	[0.92,0.93]
$C(\alpha)$	]0.92,0.93]	[0.92,0.93]

Figure 1. Asymptotic and Monte Carlo Power functions, Wald and LR tests

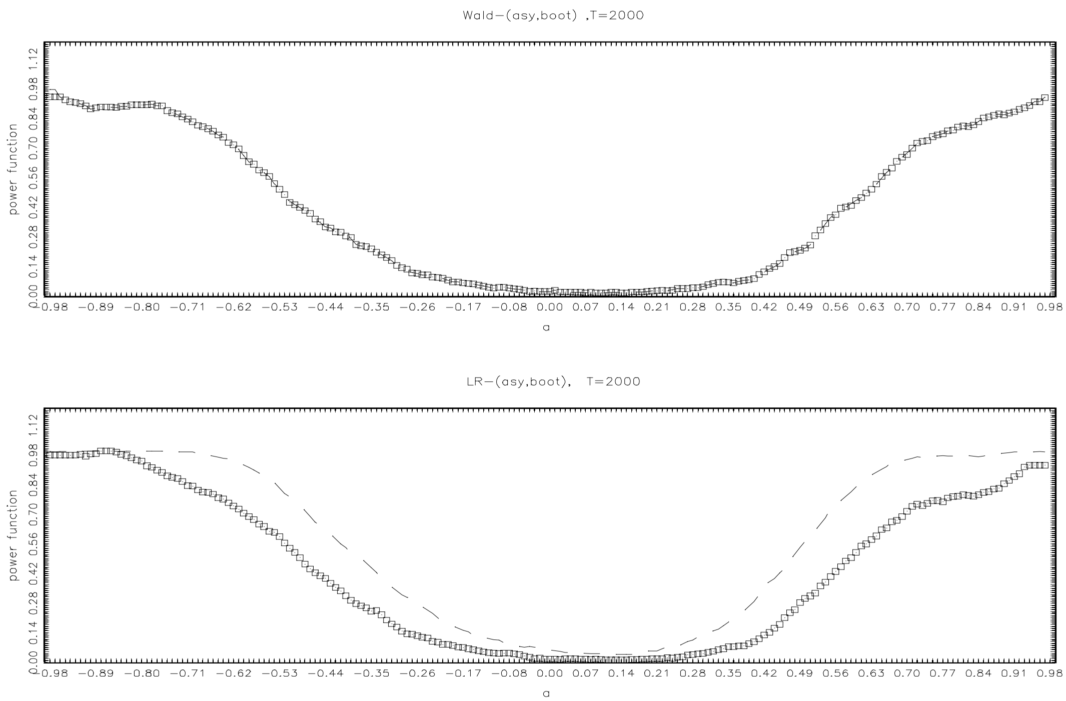
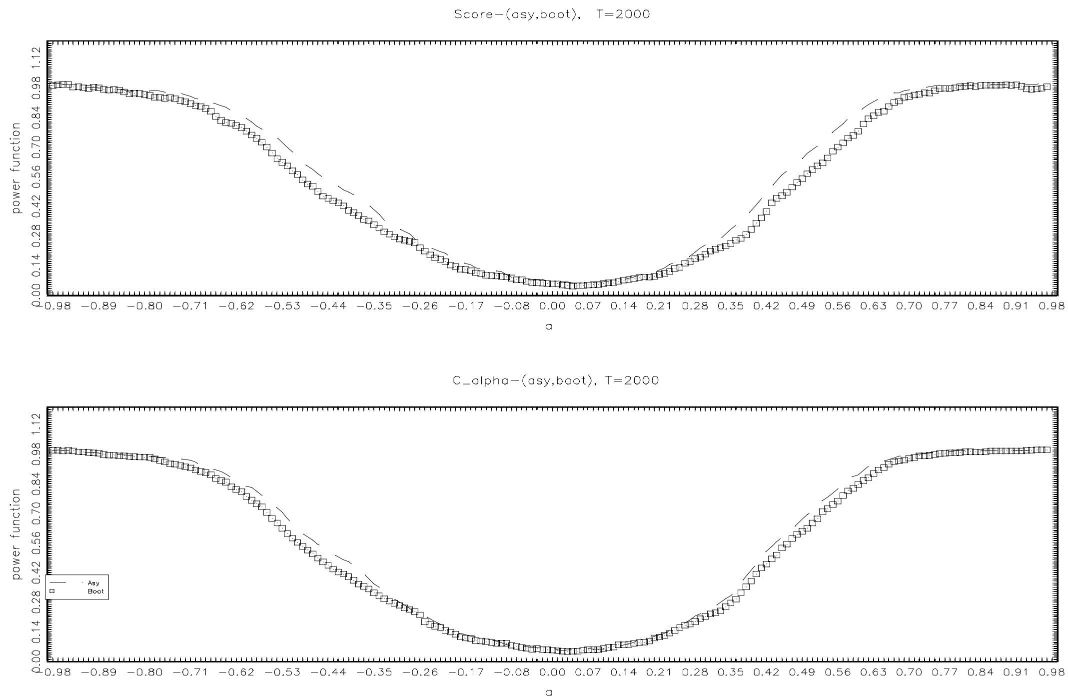


Figure 2. Asymptotic and Monte Carlo Power functions, score and $C(\alpha)$ tests



References

Alexander, C. (2001): "A Primer on Orthogonal GARCH Model," Discussion paper, ISMA Center.

Andersen, T., G. (1994): "Stochastic Autoregressive Volatility: A Framework for Volatility Modelling.," *Mathematical Finance*, 4(2), 75–102.

Andersen, T., and B. Sørensen (1996): "GMM Estimation of a Stochastic Volatility Model: A Monte Carlo Study," *Journal of Business and Economic Statistics*, 14(3), 328–352.

Andersen, T. G., H.-J. Chung, and B. E. Sørensen (1999): "Efficient Method of Moments Estimation of a Stochastic Volatility Model: A Monte Carlo Study," *Journal of Econometrics*, 91, 61–87.

Andrews, D. W. K. (1987): "Asymptotic Results for Generalized Wald Tests," *Econometric Theory*, 3, 348–358.

——— (2001): "Testing When a Parameter is on the Boundary of the Maintained Hypothesis," *Econometrica*, 69(3), 683–734.

Baillie, R., and H. Chung (2001): "Estimation of GARCH Models from the Autocorrelations of the Squares of a Process," *Journal of Time Series mAnalysis*, 22(6), 631–650.

Baillie, R. T., T. Bollerslev, and H. O. Mikkelsen (1996): "Fractionally integrated generalized autoregressive conditional heteroskedasticity," *Journal of Econometrics*, 74, 3–30.

Bansal, R., A. R. Gallant, R. Hussey, and G. E. Tauchen (1995): "Nonparametric Estimation of Structural Models for High-Frequency Currency Market Data," *Journal of Econometrics*, 66, 251–287.

Barnard, G. A. (1963): "Comment on "The Spectral Analysis of Point Processes" by M. S. Bartlett," *Journal of the Royal Statistical Society, Series B*, 25, 294.

Barndorff-Nielsen, O. E., and N. Shephard (2002): "Econometric analysis of realised volatility and its use in stochastic volatility models," *Journal of the Royal Statistical Society, series B*, 63, 253–280.

——— (2003): "Realised power variation and stochastic volatility models," *Bernoulli*, 9, 243–265 and 1109–1111.

Berger, A., and S. Wallenstein (1989): "On the Theory of $C(\alpha)$ tests," *Statistics and Probability Lettres*, 7, 419–424.

BIRNBAUM, Z. W. (1974): "Computers and Unconventional Test-Statistics," in *Reliability and Biometry*, ed. by F. Proschan, and R. J. Serfling, pp. 441–458. SIAM, Philadelphia, PA.

BOLLERSLEV, T. (1986): "Generalized Autoregressive Conditional Heteroscedasticity," *Journal of Econometrics*, 51, 307–327.

BOLLERSLEV, T., AND H. O. MIKKELSEN (1996): "Modelling and Pricing long memory in stock market volatility," *Journal of Econometrics*, 73, 151–184.

BOLLERSLEV, T., AND J. H. WRIGHT (2000): "Semiparametric estimation of long memory volatility dependencies: The role of high-frequency data," *Journal of econometrics*, 98, 81–106.

BREIDT, F. J., N. CRATO, AND P. LIMA (1998): "The Detection and Estimation of Long Memory in Stochastic Volatility," *Journal of Econometrics*, 83, 325–348.

BROCKWELL, P. J., AND R. A. DAVIS (1991): *Time Series: Theory and Methods*. Springer, New York, 2 edn.

CHIB, S., S. KIM, AND N. SHEPHARD (1998): "Stochastic volatility: Likelihood inference and comparison with ARCH models," *Review of Economic studies*, 65, 361–393.

CHIB, S., F. NARDARI, AND N. SHEPHARD (2002): "Markov Chain Monte Carlo methods for stochastic volatility models," *Journal of Econometrics, series B*, 108, 281–316.

COMTE, F., L. COUTIN, AND E. RENAULT (2003): "Affine Fractional Stochastic Volatility Models with Application to Option Pricing," Discussion paper, Université de Montréal.

COMTE, F., AND E. RENAULT (1996): "Long Memory Continuous Time Models," *Journal of Econometrics*, 73, 101–149.

——— (1998): "Long memory in continuous time stochastic volatility models," *Mathematical Finance*, 8, 291–323.

DANIELSSON, J. (1994): "Stochastic Volatility in Asset Prices: Estimation with Simulated Maximum Likelihood," *Journal of Econometrics*, 61, 375–400.

DANIELSSON, J., AND J.-F. RICHARD (1993): "Accelerated Gaussian Importance Sampler with Application to Dynamic Latent Variable Models," *Journal of Applied Econometrics*, 8, S153–S173.

DING, Z., C. W. J. GRANGER, AND R. F. ENGLE (1993): "A Long Memory Property of Stock Market Returns and a New Model," *Journal of Empirical Finance*, 1, 83–108.

DUFFIE, D., AND K. J. SINGLETON (1993): "Simulated Moments Estimation of Markov Models of Asset Prices," *Econometrica*, 61, 929–952.

DUFOUR, J.-M. (1997): "Some Impossibility Theorems in Econometrics, with Applications to Structural and Dynamic Models," *Econometrica*, 65, 1365–1389.

——— (2002): "Monte Carlo Tests with Nuisance Parameters: A General Approach to Finite-Sample Inference and Nonstandard Asymptotics in Econometrics," *Journal of Econometrics, forthcoming.*

DUFOUR, J.-M., L. KHALAF, J.-T. BERNARD, AND I. GENEST (2004): "Simulation-Based Finite-Sample Tests for Heteroskedasticity and ARCH effects," *Journal of Econometrics*, 122, 317–347.

DUFOUR, J.-M., AND A. TROGNON (2001): "Invariant tests based on M-estimators, estimating functions, and the generalized method of moments," Discussion paper, CIREQ, University of Montreal.

DUFOUR, J.-M., AND P. VALÉRY (2004): "On a simple closed-form estimator for a stochastic volatility model," Discussion paper, Université de Montréal.

DURHAM, G. B. (2004a): "Monte Carlo Methods for Estimating, Smoothing, and Filtering One- and Two-Factor Stochastic Volatility Models," Discussion paper, University Iowa.

——— (2004b): "SV Mixture Models with Application to S&P 500 Index Returns," Discussion paper, University Iowa.

DWASS, M. (1957): "Modified Randomization Tests for Nonparametric Hypotheses," *Annals of Mathematical Statistics*, 28, 181–187.

ENGLE, R. (1984): "Multivariate GARCH Factor Structures- Cointegration in Variance," Discussion paper, University of San Diego.

ENGLE, R., V. NG, AND M. ROTHSCHILD (1990): "Asset Pricing with a Factor ARCH Covariance Structure," *Journal of Econometrics*, 5, 213–238.

ENGLE, R. F. (1982): "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation," *Econometrica*, 50, 987–1007.

ERAKER, B., M. JOHANNES, AND N. POLSON (2003): "The Impact of Jumps in Equity Index Volatility and returns," *Journal of Finance, forthcoming.*

FIORENTINI, G., E. SENTANA, AND N. SHEPHARD (2004): "Likelihood-based estimation of latent generalized ARCH structures," *Econometrica, forthcoming.*

FOUTZ, R. V. (1980): "A Method for Constructing Exact Tests from Test Statistics that have Unknown Null Distributions," *Journal of Statistical Computation and Simulation*, 10, 187–193.

GALLANT, A., D. HSIEH, AND G. TAUCHEN (1997): "Estimation of Stochastic Volatility Models with Diagnostics," *Journal of Econometrics*, 81(1), 159–192.

Gallant, A. R., and G. Tauchen (1996): "Which Moments to Match?," *Econometric Theory*, 12, 657–681.

Ghysels, E., A. Harvey, and E. Renault (1996): "Stochastic Volatility," in *Statistical Methods in Finance*, ed. by C. Rao, and G. Maddala, Holland. North-Holland.

Goffe, W. L., G. D. Ferrier, and J. Rogers (1994): "Global Optimization of Statistical Functions with Simulated Annealing," *Journal of Econometrics*, 60, 65–99.

Gouriéroux, C. (1997): *ARCH Models and Financial Applications*, Springer Series in Statistics. Springer-Verlag, New York.

Gouriéroux, C., and A. Monfort (1995): *Simulation Based Econometric Methods*, CORE Lecture Series. CORE Foundation, Louvain-la-Neuve.

Gouriéroux, C., A. Monfort, and E. Renault (1993): "Indirect Inference," *Journal of Applied Econometrics*, S8, 85–118.

Granger, C. W. J., and R. Joyeux (1980): "An Introduction to Long-Memory Time Series Models and Fractional Differencing," *Journal of Time Series Analysis*, 1, 15–29.

Hamilton, J. (1994): *Time Series Analysis*. Princeton University Press Inc., New Jersey.

Hansen, L. P. (1982): "Large Sample Properties of Generalized Method of Moments Estimators," *Econometrica*, 50, 1029–1054.

Harvey, A., E. Ruiz, and N. Shephard (1994): "Multivariate Stochastic Variance Models," *Review of Economic Studies*, 61, 247–264.

Harvey, A., and N. Shephard (1996): "Estimation of an Asymmetric Stochastic Volatility Model for Asset Returns," *Journal of Business and Economic Statistics*, 14, 429–434.

Hosking, J. (1981): "Fractional Differencing," *Biometrika*, 68, 165–186.

Jacquier, E., N. Polson, and P. Rossi (1994): "Bayesian Analysis of Stochastic Volatility Models (with discussion)," *Journal of Economics and Business Statistics*, 12, 371–417.

Kim, S., and N. Shephard (1994): "Stochastic Volatility: Optimal Likelihood Inference and Comparison with ARCH Models," Discussion paper, Nuffield College, Oxford, unpublished paper.

Kocherlakota, S., and K. Kocherlakota (1991): "Neyman's $C(\alpha)$ test and Rao's efficient score test for composite hypotheses," *Statistics and Probability Lettres*, 11, 491–493.

Lanne, M., and P. Saikkonen (2002): "GO-GARCH: A Multivariate Generalized Orthogonal Factor GARCH Model," *Journal of Applied Econometrics*, 17(5), 549–564.

MAHIEU, R., AND P. SCHOTMAN (1998): "An Empirical application of stochastic volatility models," *Journal of Applied Econometrics*, 13, 333–360.

MARRIOTT, F. H. C. (1979): "Barnard's Monte Carlo Tests: How Many Simulations?," *Applied Statistics*, 28, 75–77.

MEDDAHI, N. (2002): "Moments of Continuous Time Stochastic volatility models," Discussion paper, Université de Montréal.

MELINO, A., AND S. TURNBULL (1990): "Pricing Foreign Currency Options with Stochastic Volatility," *Journal of Econometrics*, 45, 239–265.

MONFARDINI, C. (1998): "Estimating Stochastic volatility Models through Indirect Inference," *The Econometrics journal*, 1(1).

NELSON, D. (1988): "Times series behavior of stock market volatility and returns," unpublished Ph.P dissertation, Massachusetts Institute of technology, Economics Dept.

NEWEY, W. K., AND K. D. WEST (1987a): "Hypothesis Testing with Efficient Method of Moments Estimators," *International Economic Review*, 28, 777–787.

——— (1987b): "A Simple, Positive Semi-Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix," *Econometrica*, 55, 703–708.

NEYMAN, J. (1959): "Optimal Asymptotic Tests of Composite Statistical hypotheses," in *Probability and Statistics, the Harald Cramer Volume*, ed. by U. Grenander, pp. 213–234. Uppsala: Almqvist and Wiksell.

OHANISSIAN, A., J. R. RUSSELL, AND R. S. TSAY (2003): "True or Spurious Long Memory in Volatility: Does it Matter for Pricing Option," Discussion paper, University of Chicago.

QUINTOS, C. (2005): "Rank Tests for a Generalized OGARCH System," Discussion paper, University of Rochester.

RONCHETTI, E. (1987): "Robust $C(\alpha)$-type tests for linear models," *Sankhya: The Indian journal of Statistics*, 49(Series A), 1–16.

RUIZ, E. (1994): "Quasi-maximum likelihood estimation of stochastic variance models," *Journal of Econometrics*, 63, 284–306.

SHEPHARD, N. (1996): "Statistical Aspects of ARCH and Stochastic Volatility," in *Time Series Models in Econometrics, Finance and Other Fields*, ed. by D. Cox, O. Barndorff-Nielsen, and D. Hinkley. Chapman & Hall, London.

TAUCHEN, G. (1997): "New Minimum Chi-Square Methods in Empirical Finance," *Advances in Econometrics, Seventh World Congress, ed. by David Kreps, and Kenneth F. Wallis. Cambridge University Press.*

Taylor, S. (1986): *Modelling Financial Time Series*. John Wiley, Chichester.

——— (1994): "Modelling Stochastic Volatility," *Mathematical Finance*, 4, 183–204.

Wong, C. (2002a): "Estimating stochastic volatility:new method and comparison," Discussion paper, University of Oxford, Ph.d. Thesis.

——— (2002b): "The MCMC method for an extended stochastic volatility model," Discussion paper, University of Oxford, Ph.d. Thesis.